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ABSTRACT

A major problem in higher education today is that of providing educational programs for students of diversified educational backgrounds. One possible solution is to use the systems approach in the basic studies of general education to provide instructional patterns that can be accommodated to the individual needs of all students. This study set out to evaluate the feasibility of such an approach by developing model courses in mathematics and literature. The models were used experimentally in the summer of 1971. Students began with pretests which measured their knowledge in each course; then they were directed to and through the instructional units they needed. The results show that students accepted the plan enthusiastically, that most students not only finished the courses but finished with generally higher scores than are made in typical beginning courses, that the system is so economically efficient that average course costs were less than $13 a credit hour, and that instructors were able to devote more time to individual needs. (Author/HS)

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Oklahoma City, Oklahoma 73111

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EDUCATION & WELFARE
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CONTENTS

Page

Summary ii

Introduction 1

Research Method: Investigation 4

Research Method: The Mathematics Model 7

Research Method: The Literature Model 14

Conclusion 23

Reference 26

Appendices 30

A Feasibility Study for Individualized Instruction of College Freshmen in Two Basic Areas of General Education

Summary. One of the major problems of higher education is meeting the needs of the growing college population which has been drawn from every segment of society. Differences in ability and background have always created problems in structured learning situations, but the need for solutions has become more critical as higher education has emphasized more service for disadvantaged and minority groups.

One possible solution is to use the systems approach in the basic studies of general education to provide instructional patterns which can be accommodated to the individual needs of all students, those who are more advanced as well as those who are educationally deprived. This study set out to evaluate the feasibility of such an approach by developing model courses in mathematics and literature. The research team studied generally accepted standards for first college courses in both these areas. Then they created tests to measure student knowledge of these standard courses. Finally, they developed instructional units to give students essential information to meet the standard. All the materials were broken into relatively small segments so that students could be directed into only those areas where they need work.

The models were used experimentally in the summer of 1971. Students began with pre-tests which measured their knowledge in each unit of each course. Students were then directed to and through the instructional units which they needed. It was possible for a student to receive advanced standing credit; it was also possible for students to work on the course for more than one term if they needed to. The results showed that students accepted the plan enthusiastically, that most students not only finished the courses but finished with generally higher scores than are made typical beginning courses, that the system is so economically efficient that average course costs were less than thirteen dollars ($13) per credit hour, and that instructors were able to devote more time to individual needs.

The plan of meeting individual differences through the systems approach, therefore, offers a possible solution to the increasing differences of students in higher education. The plan needs additional testing and development, but initial research shows its feasibility.

INTRODUCTION

One of the great problems of present-day higher education is meeting the individual needs of students just starting to college. The problem has always been present, for there have always been different degrees of preparation among those coming to campuses. Now, however, with a steadily increasing social mobility, with a growing effort to get more minority group members into and through college, with a growing divergence of quality in secondary education, and with a national emphasis on bringing an ever larger percentage of the high school graduates to college, this problem of meeting individual needs has been greatly accentuated. Now a wider and wider range of backgrounds and abilities exists among those starting in American colleges and universities.

Unfortunately, the design of most college programs increases rather than decreases this gap between students as they proceed through their courses. Since the traditional design of college courses requires students to proceed at about the same pace, those who start behind for lack of adequate preparation generally fall further and further behind. Eventually, since no provision has been made for their special needs, those with inadequate backgrounds usually fail or do poorly enough that either they are excluded by their sense of personal frustration and failure or they are forced out by grade requirements for readmission. Present approaches also bring unfortunate consequences to the advanced students. They are commonly asked to spend considerable time on materials they already know and, not being challenged, they fail to make significant progress. The testing programs for advanced standing provide some relief, but the all or nothing at all situation is not good academically, since even well-prepared students often need information or concepts which they may not have even though they make a minimum score for advanced standing credit. Students who fail an advanced standing examination by a few points must obviously review much information which they already possess.

As an attempt to meet the needs of varied and divergent educational preparation of beginning college freshmen, a research team at Oklahoma Christian College began work on a plan which has apparently not been tried previously. Using general education guidelines for Oklahoma Christian College as set forth in a planning report (Design for Progress), the team fixed on eight crucial areas of college study: science, mathematics, government, and economics, literature, composition, history, Bible, and orientation to college life. Because fundamental knowledge and skills in these eight areas are essential to a student's conceptualization and personal expression, because these courses traditionally have high failure rates or low interest level resulting from materials' being repetitious of information from advanced high school courses, and because numerous other college courses are dependent on the principles and information of these first courses, beginning college courses in these areas doubtless play a vital role in shaping the student's attitude toward learning and in his real success with college studies.

The general plan for these courses would be, first, to measure a student's exact status in these eight basic areas at the beginning of his college career. Knowledge of these eight basic areas will be evaluated by describing the performance expected of the college student at an elementary level of each discipline. The performance objectives will be at the college and not the high school level. Information gained by the initial evaluation on these objectives will indicate the extent to which each student either falls below the goal or exceeds it. Those who exceed any specific objective will be given advanced standing credit for it and be allowed to pass on to other work. On the other hand, those who fall below the requirements of any objective will be given opportunities and instruction to satisfy the requirements.

Next, a series of learning programs should be developed for each objective so that students who do not show attainment of a given objective on the pre-test, would have an individualized program to follow for reaching it. Those learning programs will afford each student the opportunity to work in his own way and at his own speed. Certain requirements will, of course, be built into the program to insure that students are working satisfactorily and that they are receiving any personal assistance they need. As students reach the required levels, they will complete that portion of the program and will move on to something else.

Because this plan is new, because it seems to hold so much promise in helping solve the problems of divergent background and preparation of beginning college students, and because the plan requires large expenditures of money to develop the testing and instructional program, the research team set out to study the feasibility of the plan, using literature and mathematics as models. Both of these areas are concerned with somewhat difficult concepts, and both are areas in which beginning college students vary greatly in background, experience, and proficiency. Furthermore, the educational objectives of these two disciplines almost represent the poles of cognitive and affective experiences. Mathematics is quantitative and involves problem solving through analysis and cognitive processes. Literature, on the other hand, is imaginative and involves personal response that is both rational and emotional. Thus, these two basic areas represent different kinds of teaching and learning problems and therefore provide excellent models for evaluating the feasibility of this program for assisting beginning college freshmen.

The research team spent one year formulating objectives and developing tests and materials related to those objectives for the literature and mathematics models. Members of the research team visited other colleges and universities, corresponded with professors who have experimented with improving instruction or curriculum, investigated programs designed to accommodate different ability groups, and consulted with experts. From these investigations came the statements of objectives for both models. The team then developed the instruments to test for attainment of the objectives at the same time it began developing instructional materials to help students attain the objectives in English and mathematics.

The models were then used with incoming freshmen students at Oklahoma Christian College, an especially good site for this study. The College has an open admissions policy for any high school graduate whose character references are acceptable. The College has also been deeply involved with experimental use of different media to accomplish learning goals. The Mabee Learning Center has encouraged the use of taped instruction as well as other forms of instruction that can be specially modified for the needs of individual students. Finally, because the whole college community of Oklahoma Christian has become accustomed to an extremely flexible pattern of class meetings and instructional design, this program for individualized education was well received by students and non-participating faculty members.

By formulating objectives for these two models, studying experimentally the instructional materials for these two courses, and studying costs and results of an experimental use of the program the research team has concluded that this general plan for accommodating individual needs of beginning college students is educationally sound and economically feasible. The objectives and models developed as part of this study need revision and further refinement, without question, but they are useful guides in the first stages of seeking solutions to higher education's increasingly important problem of serving college students with differing degrees of preparation.

This report of research to determine the feasibility of special instructional patterns to accommodate the varied preparation and abilities of beginning college students describes (1) the research team's investigation of other experiments and research relating to the proposed design, (2) the rationale, development, and utilization of the instructional models in literature and mathematics, and (3) the findings and conclusions of the study.

RESEARCH METHOD: INVESTIGATION

The research team began by investigating the experiments and designs of creative teaching in mathematics and literature. The primary purpose of this search was to discover techniques, approaches, materials, and theories which would be helpful in developing effective instructional models. This investigation did provide a wealth of ideas and hints for improving college instruction.

Beginning with a GIPSY/ERIC search* and an investigation of education and professional journals for the past five years, the research team found that no published research has experimented with a plan like the one this project was analyzing and investigating for feasibility (cf. Morgan). However, because the plan incorporates elements of testing programs for college advanced standing and instructional programs to help students remedy specific deficiencies in special disciplines, research and experimentation in both these approaches were useful in suggesting theories and techniques. The works of Professors Godshalk, Swineford, and Coffman served as a model for evaluating developmental skills, as did the findings of Professors Cooley and Klopfer for cognitive preception and retention.

One area on which the early research concentrated was that of determining what comprised the essential knowledge of beginning courses in literature and mathematics. In mathematics much work has been done in establishing content for basic courses. Studies of high school programs, which usually reflect what the secondary schools think higher education will be teaching (Kruglak and Belasco), reviews of basic curriculum (Sobel and Cooperative General Science Project), and experiments with course content (Collagan and Connelly) indicated that a certain unanimity exists concerning essential information and concepts in mathematics courses. In literature the diversity of content and the different views of what constitutes essential concepts are reflected in the reports from the Dartmouth Conference by Muller and Dixon. Squire and Appleby, examining high school English curricula, showed the broad range of requirements and philosophy, although Professor Ballit, interprets the same curricula as amazingly uniform in their primary expectations.

Study of existing college programs of instruction yielded a variety of results for remedial, advanced, and regular students (Woodby, Pooley, Collagan). The usual approach taken in larger colleges and universities is to provide different courses for different backgrounds or different majors. For remedial students in mathematics there is usually some type of high school algebra course and in literature there are usually simplified readings connected to a remedial composition program. In some institutions these courses do not satisfy any requirements, even electives, for a degree program. Very little seems to have been done in motivating these students to work for a higher level of achievement.

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*The question presented to ERIC was, "What is known about individualized or personalized instruction for a general education mathematics course at the college level for both disadvantaged and non-disadvantaged freshmen?"

For the advanced student in mathematics there is usually an honors calculus program, a speeded up course, which has not been completely successful, according to most authorities: this honors course is designed mostly for science and mathematics majors with very little thought for other students. For advanced students in literature, the typical honors course studies the classics or other works, using formalistic critical approaches. Goals commonly stress the quantity of reading rather than the quality of reading experience. For the average mathematics student what course he takes depends upon his major. There are sequences of courses for elementary education majors, business majors, mathematics majors, engineering majors, etc. These courses often have different titles and numbers, but many times the contents and approaches make the courses indistinguishable. While it is true that different objectives are present in these various sequences of courses, it is also true that a fundamental body of mathematics knowledge is common to all the sequences. For the average student beginning college only one literature sequence is available--usually as part of the composition program. Second year literature courses often give students options among national survey sequences.

The alternative to multiple courses is to offer fewer sequences, thus concentrating either the main body of knowledge common to all the various disciplines which use mathematics or the skills and concepts which can make literature a meaningful experience, aesthetically and intellectually. Such an alternative makes fewer provisions for special abilities or disabilities and for specialized interests. Smaller colleges and universities normally are forced by economics into this alternative.

The research team discovered that instructional techniques vary greatly and that little has been done to determine the long-term effects of any of the techniques. Most of the research has considered only short-term cognitive effects. The techniques fall generally into the following categories: lecture, lecture-discussion, audio-tutorial, video-tutorial, programmed instruction, and various combinations of these. The traditional lecture method is still the most widely used one. It is effective whenever there is not too much variation in the background or motivations of students. It can provide for good harmony between student and teacher. On the other hand, the lecture-discussion technique seems to be used for large classes in an effort to save teacher time and money while retaining the advantages of the lecture method. Since the students usually cannot ask as many questions during lecture, the smaller discussion group permits opportunities for questions and interaction among students and instructor. Discussions sometimes seem unrelated to the lecture, for rarely does the lecturer direct the discussion session.

The audio-tutorial approach involves some type of audio-tape materials along with a workbook and/or response sheets (Duffy). It is used as a teacher substitute at times as well as a supplement to the class (Moon and Davis). At OCC, for example a three hour mathematics course using this method met twice a week for a lecture period then once a week for a tape in conjunction with a response sheet (McMillan and Brown). This method is much like a programmed textbook except the program is on the tape. It can be an effective method but requires a great deal of effort in preparing the tapes. Students also seem to feel that is less personal. Video-tapes have been used to some extent but the cost and time of preparation are major limiting factors (Jobe). Video lectures are also used in some colleges where a teacher lectures before a TV camera and it is transmitted to several classrooms with graduate assistants in charge. This method seems to have a limiting factor in that the teacher is out of touch with the student. They take the appearance of being much like regular television classes which are certainly effective for motivated students but do not provide much help otherwise. Programmed materials have been used extensively in the last few years. They proved to be quite effective in some instances (Collogan; Lange) while more traditional methods showed better learning results in others (Rokeach and others; Machetanz and others). In general they are effective whenever the student follows instructions and has some degree of motivation. Without motivation and sincere effort in following the set procedure for the materials it is doubtful whether or not they are as effective as other methods. There is some indication that they make good supplements to the classroom rather than replacement of standard materials (Morrow). Much of the programmed materials appears to be quite good and some even allows the students to skip parts which they already know by taking pre-tests over each section of the book (Eraut).

To sum up the results of the preliminary research phase, no one instructional program or technique seems to be championed above all others. Each type of program and instructional technique has certain advantages and disadvantages which must be reckoned with before it is implemented. No plan like the current investigation has been or is being carried out. The following sections describe the models into which the germinal ideas of the initial research phase were incorporated. Both the mathematics model and the literature model are described in detail, and it was through the development and utilization of these models that the feasibility of this plan for individualizing instruction to accommodate the differing abilities of beginning college freshmen has been determined.

RESEARCH METHOD: THE MATHEMATICS MODEL

To determine the needs and objectives in mathematics at the elementary level in general education the advice and assistance of consultants provided the most valuable aid. The advisors for the project were Dr. Max Sobel, Montclair State College; Dr. Thomas Hill, the University of Oklahoma; Dr. Sarah Burkhart, Tulsa Public Schools; Dr. John Jobe and Dr. Gerald Goff, both of Oklahoma State University. The consultants were of special help in developing the rationale for general education mathematics courses and in deciding what areas deserved priority in this model of a basic studies course. The following account of the methods used in this experimental development considers Rationale and Objectives, Instructional Design, and Experimental Application.

Rationale and Objectives. The pragmatic philosophy behind this model is that mathematics at the elementary level in general education should have the following two main objectives:

1. To provide the student with the proper mathematical tools needed to function at an adequate level in present day society.

2. To provide the student with some appreciation of these mathematical tools by showing some elementary applications of mathematics at a practical level.

These two objectives should and do complement each other, for without the proper mathematical tools students cannot understand much less appreciate the usefulness of mathematics at the practical level. Also without some knowledge of the usefulness of mathematics in various applications students have difficulty knowing why certain mathematical concepts must be learned. Too often in the past mathematics has been presented from the algebraic viewpoint of manipulation of rules without any practical problems. Even when "practical" problems were presented, they were much like this example: "Tom is three times as old as Sue. In 5 years he will only be twice as old as she is. How old are they?" The same teaching as above could be done with a problem most of us face, such as, "Two cans of a certain product are on sale for 79 cents. Each can contains 7 ounces. The same product can be bought in a 10 ounce can for 54 cents. Should one buy the cans on sale?"

With the above objectives in mind full concentration was focused on the material to be used in the course. In order to provide the student with the proper mathematical tools it was thought that he should know and understand the operations on the counting numbers, integers, and rational numbers. This would include manipulation of fractions and decimal notation. Next, he should be able to solve simple equations, especially those of the type usually encountered in elementary chemistry, biology, psychology and business courses as well as those encountered in every day experiences such as balancing a bank account, filling out income tax forms, calculating interest and so forth. This would involve manipulation on and with algebraic expressions, the solving of simple word problems, the basic rules of exponents, and approximation of roots. Even though this is certainly not an exhaustive treatment of arithmetic or algebra it does comprise the essentials of mathematical knowledge that is necessary for a student to function in present society (Raygor and Wallace).

The basic objectives for these areas are embodied in the following statements:

Objectives for Structure of the Number System

1. The student can distinguish between counting numbers, integers, rational numbers and real numbers.
2. The student can define a binary operation on a set.
3. The student can show an example of the distributive law of multiplication over addition.
4. Given a set S and a binary operation the student can determine whether or not the set is closed, commutative, associative, has an identity or has inverses.

Objectives for Algebra

1. The student can do the rules of sign for the various operations of addition, subtraction, multiplication, and division.
2. The student can simplify algebraic expressions.
3. The student can determine whether or not two fractions are equivalent.
4. The student can use the basic properties of exponents such as,

   $a^m \cdot a^n = a^{m+n}$, $(ab)^m = a^m b^m$, $(a^m)^n = a^{mn}$, $a^m \div a^n = a^{m-n}$.

5. The student can solve and check an equation for the variable indicated.
6. The student can set up the relationship among variables from simple verbal problems.

Objectives for Geometry

1. The student can state the Pythagorean Theorem for right triangles.
2. The student can solve for the third side when two sides of a right triangle are given.

3. The student can give at least one intuitive demonstration of the truth of the Pythagorean Theorem.

4. The student can construct the measure of the square root of a counting number by using Pythagorean triangles.

5. The student can approximate the square root of a positive integer correct to one decimal place.

Objectives for Statistics

1. The student can find the arithmetic mean, mode, and median of a set of data.

2. Given a set of data the student can interpret the data and select the correct average to use.

3. The student can find the range and standard deviation of a set of data.

4. The student can explain what the standard deviation of a set of data indicates.

Objectives for Probability

1. The student can define outcomes, sample space probability, event random, union of sets, intersection of sets, mutually exclusive events, independent and dependent events.

2. Given an event the student can find the probability of this event after defining the sample space.

3. The student can write the formula for the probability of mutually exclusive events.

To provide the student with some appreciation of the mathematical skills described in the objectives it was felt that some simple applications of how mathematics is used would not only teach him the mechanics of manipulation but also give him an understanding of the importance and usefulness of the subject. The contents of the actual applications themselves are not as important as the spirit of mathematics which they present (Jewitt and Others). The first application combines a brief history of the Pythagoreans and their achievements with the famous theorem on right triangles. The second application concerns statistics with calculation of the measures of central tendency and measures of variation. An attempt is made to show the need for these types of measures and how to interpret them. The third area of application concerns probability. This area teaches students to consider all of the possible outcomes of a given action and to make judgments in areas of uncertainty. An excellent source book for determining what should be taught at the elementary level is the Committee on the Undergraduate Program in Mathematics' publication, A Course in Basic Mathematics for Colleges (Jewitt and Others). Although this publication was not available in time to help in formulating the material used in this study it does verify the material as that which should be taught in the colleges at the elementary level.

Instructional Design. The development of a model for mathematics was formulated as an extension of the so called mini-course concept. This concept seemed to have the necessary flexibility to serve differing educational backgrounds of beginning students. As the model was conceived it was to have the following characteristics:

1. It should achieve the two main objectives and the mathematical objectives listed earlier in this report;

2. It should provide for the poorly prepared student by giving him sufficient time and opportunity to proceed at a pace suited to his needs;

3. It should provide for advanced students by giving them "credit by examination" for areas they already know;

4. It should provide for the bulk of regular students who need work in one or more basic areas to make up these deficiencies without being required to spend time in areas where they are adequately prepared;

5. It should serve a large number of students but give them the same advantages as those students in small classroom situations.

The model works the following way. At the beginning of the trimester each student takes a pre-test over five areas of mathematics: structure of the number system, algebra, geometry, statistics, and probability. (See Appendix J). He then receives an individualized report telling him those areas which he passed and those areas in which he was found to be deficient (See Appendix K). If he was deficient in none of the five areas then he has finished the course and receives a grade of A. If he is deficient in only one area he has one of two options. He may take a grade of B for the course and do nothing or he may complete the section in which he was found to be deficient at the 80% level or above and receive a grade of A for the course. If he is deficient in 2, 3, or 4 areas he has the following options: he can complete all deficient sections at the 80% or above and receive a grade of A for the course; he can complete all but one of the deficient sections at the 80% level or above and receive a grade of B for the course; he can complete all but one of the deficient sections at the 70% or above and receive a grade of C for the course; if he fails to achieve the above goals he receives no credit for the course; a grade of F is given.

A student found to be deficient in all five areas is advised to take one of the following two options depending upon the results of the pre-test. If his score on the pre-test indicates that he may profit from attending regular sessions of the course, he is advised to do this until he has completed a minimum of four sections at the 70% level or above. If his score on the pre-test indicates that he needs to proceed at a slower pace than regular students he is then advised to purchase a programmed textbook (Michael Eraut, Fundamentals of Arithmetic) and attend special classroom meetings with an advanced mathematics student in charge. These are small classes and the student tutor does not lecture to them. Instead he is used to provide motivation and guidance in completing the programmed textbook. The student is encouraged to proceed at a rate suitable to his needs but not allowed just to sit in the course. After completion of this workbook he takes a test. If he passes the test at the 70% level or above, he has then completed the structure and algebra portions of the course and is ready for the sections on geometry, statistics, and probability. If he does not have time to complete these sections before the end of the trimester, he receives a grade of W for the course and must re-enroll another trimester for final completion of the course. If he does complete the course, he is graded as the other students. If he does not pass the test over the workbook or if he has not completed the workbook by the end of the trimester but is making satisfactory progress towards its completion, he is given a grade of W for the course and must re-enroll another trimester for final completion.

To complete a section in the regular sessions two ways are available. The first way is to attend classes over the deficient section as given by the class outline (See Appendix L) and pass the test over this section. A class schedule by date and subject to be covered is given each student at the beginning of the trimester. The second way to complete a section is for the student to work through each lesson of the section on his own and complete the problem sets at the end of each lesson (See Appendix M). Then after demonstrating to the instructor or tutor that he has done this, he will be given an examination over the section. If he passes the test, he then has completed the section. If he fails the test he then can work on the material on his own until he is believed ready for another examination, or he can take the regularly scheduled classes over the section as given by the class outline. Usually one week is the minimum time between examinations over a given section. Also in order to complete a section without attending regular class he must do so before regular classes are offered over this section. This requirement is added to encourage the student not to put off working on a section until too late in the trimester to complete it. Thus, this second way of completing a section of the course is an option only if the student uses it before regular classes are offered on that section. If he has not completed it before regular classes are offered, he then must attend classes to complete it.

In the event a student fails a regularly scheduled examination or does not pass it at the level he wants he may retake this test at one week intervals until he passes it or the trimester ends. He can make up one examination during the final test period but not more than one.

Experimental Application. The model was tested on forty-one students during the summer trimester, 1971. The students tested were not classified in any particular way except that they all needed a basic mathematics course to satisfy the general education requirements of the college. None were mathematics or science majors. ACT scores on the students tended to show them average or slightly below the average mathematics class at this level at OCC. The pre-test was given to 38 students of which 20 passed one or more sections of the pre-test, with geometry being the section that most students passed. The final grades of the different groups are shown below:

Final Grade	Those passing at least 1 of 5 sections on pre test (20)	Those failing all 5 parts of pre-test Remedial section (15)	Regular class (6)
A	7		1
B	8	3	1
C	3	3	1
F	2	6	2
W		3	1

The fact that more than 25% of those failing all five parts were able to make an A or B indicates that even the poorly prepared succeeded unusually well in the program.

Dr. Richard Cave, a specialist in statistics, evaluated the testing procedures to determine their validity and to assess the effectiveness of the instruction. The following is his report.

I. Evaluation of the Test Used.

Two identical tests were administered, once at the beginning of the course and again at the end of the course. The pre-test determined who would have to attend each of five sections of the course. Those students who passed a section of the test were exempt from that section of the course. The second administration of the test came at the last week of the trimester as a post-test. All students were instructed to take the post-test, though it had no bearing on the grade. There were 29 students for whom we had complete pre and post scores.

Of the five subtests, the Structure subtest (section I) was the most complex, with three questions labelled A, B, and C. A and C were matching-type and B was true false type. Question A was a five item matching question with six alternatives; typically students will get one or two of such items correct just by chance alone. The mean for the pre-test was 1.38, a chance score, and for the post-test 2.90. B was a true-false question with five items. If a student answered all of them true he would get three right and if he answered false to all of them he would get two correct. The chance score was 2.5. The mean of the pre-test was 2.52 and for the post-test 2.34. The standard deviations for pre and post was 1.21 and .86 respectively (smaller than for any other question on the test). C was a six-item matching question with nine alternatives. We would expect students to get one of the six correct just by chance. The mean for the pre-test was 3.03 and for the post-test 3.52. The correlations between A, B, and C and the total section I score were .62, .56, and .77 for the pre-test and .79, .52, and .80 which shows that question B contributed less than the other two questions.

The other four subtest were multiple choice, except for one true-false item in subtest III. In Table I the chance scores and means are given for pre and post tests.

TABLE I

Means and chance scores for subtests II through V.

Subtests	Chance Score	Mean pre	Mean post
II	1.6	3.62	4.45
III	1.5	3.17	3.90
IV	1.4	3.00	4.69
V	1.2	1.48	2.62

Only subtest V failed to correlate better than .50 with the total test score (.33 for pre-test and .45 for post-test). The correlations between subtests were all below .50 except that subtest III correlated .50 with subtest IV.

The comparisons, therefore, show that all subtests measured some general mathematical ability, and each test measured some specific knowledge not measured by any other subtest.

II. Pre and Post Comparisons.

The mean scores improved for all five sections of the test. In Table II is a summary of pre and post mean differences.

TABLE II

Mean differences for the five subtests db = 28

Subtest	Mean difference	t	Probability
I	1.83	3.46	.001
II	.83	2.06	<.05
III	.72	3.17	<.01
IV	1.69	3.48	<.001
V	1.14	3.90	<.001

Comparison between students who took a section of the course with those who were exempt from that section, was possible only for Section III. The difference between means was .189, not significant. For the other sections there were not enough people who were exempt to make any statistical comparisons. In each section those who were in the course made greater improvement than those who were exempt. The difference indicates exposure to the class was significantly beneficial.

RESEARCH METHOD: THE LITERATURE MODEL

The literature model was developed in consultation with Robert W. Daniel of Kenyon College, Charles Webb of the University of Illinois, and Robert Zoellner of Colorado State University. Each provided ideas and suggestions for the objectives, the philosophies, and the processes of the model, although none would have created a program precisely like the one which was developed. The description of the literature model considers Rationale and Objectives, Readings and Instruction, Instructional Design, and Experimental Application.

Rationale and Objectives. The literature model gives special consideration to the relationship of literature to the humanities. Fundamental to the design is the concept that students develop personally through their experience with literature and that their readings call forth emotional, intellectual, and psychological responses (Fromm). Personal understanding of these responses is as important as the background or factual content of the reading (Holland and Lesser).

The research team considered several possible approaches. The traditional national survey was ruled out because most high school curricula include basic surveys of American, English, and world literatures. Such previous experience would dull student's sense of expectancy for their initial college experience with literature. An epochal approach was considered, but rejected because of the time required to establish the philosophical, historical, and social backgrounds which are essential to clear understandings of any past epoch (Booth). Both these approach have the disadvantage of emphasizing factual information, which may or may not be relevant to developing insights into the humanities and humanistics values. The greater need in literary studies seems to lie in developing abilities to read and interpret independently (Shumake and Dixon). Students have almost come to expect that any reading assignment for an English class will require an English teacher for proper analysis and interpretation. Students need the competence and confidence to read, contemplate, and interpret literature (Dixon; Mandel). In light of this rationale, the terminal objective for the model course took this form: "Given any literary work of average complexity, a student will be able to read and analyze the piece for direct and implicit meaning, for character development, for narrative or logical structure, and for the interrelationship of all these elements."

Since the final objective of the literature program thus concentrates on the development of interpretative skills, the research team and consultants, analyzing the processes involved in interpretation and response, formulated the following statement of philosophy about the objectives of the model course:

The principal goal of this course is for students to learn to

> respond emotionally and intellectually to literature. True responses are, however, not really measurable, and we must try to find a course between the engulfing waves of emotionalism, mere sentimentality, "sweetness and light," dillettantism and the sterile stands of logical analysis, counting syllables, scholastic criticism, and pedantic exercise. These objectives, then, have been formulated with the view that many of the basic concepts are cognitive and measurable. Much of the response is, on the other hand, affective and not easily measurable. By creating situations in which these responses are verbalized and described, the program will provide information and insights which will allow formulation of measurable affective goals.

In order to employ the systems approach with the model course, these specific goals were established.

Objectives for the Methodologies of the Humanities:

1. A student can explain ways in which the humanities are different from the sciences and social sciences.

2. A student can postulate several theories about literature's relation to the other humanities.

3. A student will view any literature as a text through which intellectual and emotional experiences are available to him.

4. A student can describe several theories about the psychological effect of literature, especially insofar as those effects are related to aesthetics.

5. A student can describe his emotional and psychological responses to an imaginative work.

Objectives for Meaning:

1. A student can approach an understanding of any literary work through a study of its historical, social, cultural, political, philosophical, and autobiographical context.

2. A student can demonstrate how the contextual information sheds light on a specific work.

3. A student can interpret orally several works so that the tone, mood, conflict, and meaning are communicated to a group of students.

4. A student can recognize the principal theme and the minor themes of several assigned works.

5. A student can discriminate the different attitudes conveyed by several works treating the same theme.

Objectives for Character:

1. A student can describe at least six ways in which writers can develop characters in their works.

2. A student can analyze a work, showing all the methods used in delineating characters.

3. A student can discuss intellectual and emotional ranges of characters from selected groups of stories, poems, and plays.

4. A student can recognize differing degrees of complexity and simplicity in characters.

Objectives for Structure:

1. A student can recognize the logical or narrative basis for structure in poetry, drama, and fiction.

2. A student can describe the structure of poems, short stories, and plays.

3. A student can discuss ways in which structure is related to meaning and characterization.

Final Objective:

A student can write an analysis of a moderately difficult work which he has never seen before, taking into account context, meaning, characterization, structure, their interrelationships, and cognitive and affective responses.

Readings and Instruction. The research team decided to concentrate on American writings since 1945, for Contemporary American writing is conspicuously concerned with cultural and social issues facing the humanities. Although that same statement can be made regarding the literature of most past ages, students are not often aware that it is true unless they are provided a mass of information which allows them to become familiar with that age. Few students are not aware of the major struggles and experiments of the past twenty-five years.

The present generation has been forcefully made aware of the nature of the individual and the importance of individual liberties. The reading design, capitalizing on such awareness, concentrates on leading students into a greater sensitivity to the complex nature of man--his infinite variety and his unlimited potential to think and feel, topics students already recognize as important. The nature of the humanities seems to be most clearly defined through a consideration of the individual's relationship to other individuals and groups. For the present generation those conflicts have most often centered upon racial differences, the complex nature of a growing government which tends to involve itself more closely with all phases of the individual's life, family relations where ties have become less meaningful as lines of authority have been repeatedly blurred, and a civilization where large megalopolises have required flexibility and adaptation of which man may yet be incapable.

Focusing, then, on such timely issues as individual nature, racial conflict, governmental control, urban problems, and family relations, this model attempts to draw students into direct and personal confrontation with their own views. To bring about such involvement it is essential that the reading selections be unusually effective in holding attention and insinuating values. Therefore, all the prospective readings were selected for reader appeal and relation to the issues evaluated by a group of ten students. All the students were college sophomores; half were better than average students but were not especially interested in literature; the other half had histories of class cuttings and general neglect of academic responsibilities. Students were promised one hour of A in an independent study course if they would read and evaluate the possible selections. Each student was given a pad of evaluation sheets (See Appendix A) and was asked to fill out one sheet for each reading he was assigned. Each student read approximately eighty poems, short stories, and plays. Student preference determined the final reading selections used for the model (See Appendix B).

It was decided that instructional material needed to be available in several forms if students were to be able to progress at their own rate. Preliminary materials on humanities, texts, and contexts were presented in lecture form so that students could meet the instructor. These materials were also incorporated into printed essays or taped instruction so that students could review the materials and study them if they were absent from the lectures. Ideas about meaning, character, and structure were developed into essays, which were distributed to students along with the reading selections (See Appendix C). All the materials went through the process of empirical development with four students who were enrolled in a traditional freshman English course winter trimester, 1970-71.

Instructional Design. The development of instruction within the literature model was predicated on three premises. First, in all ways possible the course should foster in the participants a knowledge of their human potential, a respect for their individuality, and habits of continual self-evaluation. Second, the course should permit or encourage, which ever the case may require, both the formulation of personal responses and the testing of those responses through interaction with others. Third, the course should involve students in a creative effort to communicate their experiences and discoveries with their readings.

These premises gave rise to a testing program which established only a framework or set of guidelines for students' interpretations. The Pre-Test and the Achievement tests (See Appendix D) force the student to develop responses. The difficulty in using such tests is that evaluation is somewhat subjective. Subjectivity, is, however, inherent in humanities studies, and if at least two readers evaluate each test, the results will probably be reliable.

The basic premise also prompted the inclusion of "rap sessions" in the design of the program. Early in the course seven or eight students are grouped for informal discussions of the readings. An advanced student in English is assigned to each rap group, but he attends only as a spectator and writes a report of the session for the instructor (See Appendix E). Rap sessions may be visited by the instructor, but he attends only as a spectator. Each rap session is directed by a different student leader, who was selected by the group at its previous meeting.

Leaders are responsible for drawing out discussion of meaning, characterization, and structure. Leaders may consult with the instructor or the advanced student. Of course, he can rely on the general study questions which are suggested for each assignment. Whatever he does he strives to stimulate thought and exchange of ideas. After each session, the discussion leader evaluates in writing his preparation, his strategy for involving the group, and the discussion's relevance to the specific objectives for the course. Using a simple check sheet, other members of the group evaluate the discussion leader (See Appendix F).

After rap groups have completed all their assignments, each member of the group fills out an evaluation form for other members (See Appendix G), and he writes a one page evaluation of his contribution to the group's workings and thought.

A third practice to grow from the basic premise is that of requiring a performance program from each rap group. The chief objective of the performance is to give students experience in oral interpretation and public discussion of literature. Each rap session selects a program chairman who coordinates the preparation and presentation of the program. Guidelines for the performance are very general, but stress the importance of ideas, coherence, unity, and polish. All students are invited to each performance program, and each must attend and evaluate at least one (See Appendix H).

The general pattern of student processes is illustrated graphically in Appendix I, but the following description explains the model's procedures, step by step.

Initial information: Instructions for first lecture; details about diagnostic test and procedures for taking the test.

Lesson I: "Literature as a Humanities Study"*

Take diagnostic test (See Appendix D)

Read "A Worn Path" in preparation for Lesson II

Lesson II: "Literature: The Text"* Give instructions about Performance Program. (See Appendix H)

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*Appendix C.

Conference on Diagnostic test: assign to Rap Group.

Read The Crucible in preparation for Lesson III.

Lesson III: "Literature: The Context."*

Meet with Rap Groups; select first leader; study evaluation forms for leaders and entire study group (Appendix F and G).

Section of Assignment.

Rap session for meanings (each person should leave with an assignment about contexts); select next leader: select Chairman for Performance Program.

Prepare for rap session; distribute written report to members of discussion group two days before the scheduled meeting.

Rap session for meaning through context over readings; select next leader.

Rap session for discriminating meanings over readings; groups to continue will select next leader.

(Students exempt from sections on "Character" and "Structure"* will read those two essays; they will then move to Performance Preparation.)*

Prepare for rap session by reading "Character,"* Section II of Assignment.

Rap session on how authors give characters personal and psychological being (each person will select one character from readings and will report next session on the nature of the character and how he was presented to the reader); select next leader.

Rap session on characters (each member will select one reading to study and report back on how characters relate to meaning of the whole); select next leader.

Rap session on characterization and meaning.

Students exempt from work on "Structure" will read that essay; they will then move to Performance Preparation.

Prepare for rap session by reading "Structure"* and Section III, of Assignments.

Rap session on structure; select next leader.

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*Appendix C

Rap session on interrelationship of meaning, character, and structure; evaluation of participants.

Performance: each rap group will present a thirty-minute program performance section for critical evaluation. (Appendix H)

Evaluation of participants (See Achievement Test, Appendix D).

Those failing final test will have a counseling session over the test: they will have additional instruction and then take another achievement test.

Grades for the model course are based entirely on the testing and the performance program. Students who make 40 points on each of the three parts of the Pre-Test are given an A after they meet for three rap sessions and present a performance program. For all others the Achievement Test counts 75 points and the program 25. Grades are based on a scale of A, 91-100 points; B, 81-90 points; and C, 71-80. Students who make less than 71 will receive either a W or an F. The F indicates that a student has either not attended rap sessions or has demonstrated little effort in developing the skills required for the course. Students who have tried with only minimal success will receive a W, which is no discredit. They can enroll again for the course the next trimester, and work only in the section or sections where they did not attain the objectives before.

The final grades for the group were as follows:

A	17
B	10
C	29
W	4

The fact that all students completed the work, even though four did not obtain a passing score on the achievement test, indicates that the group was unusual or that the materials were especially appealing. Student reaction was unusually favorable. Students' personal evaluations of their own preparation and effectiveness were sharply critical for at least fifty of the students. On the other hand, their evaluations of each other were seemingly generous. For only eight students were the evaluations completely negative. For all the other students approximately eighty per cent of the ratings were either "Average" or "Good."

Among the sixty students in the course, thirteen were beginning college freshmen, three were second semester freshmen, fourteen were first semester sophomores; all the others were juniors and seniors. Among the thirteen beginning college freshmen, there were 5 A's, 1 B, and 7 C's--significantly, no W's. Among the second semester freshmen two made W's and one made C. The fourteen sophomores made 5 A's, 3 B's, and 6 C's. The thirty juniors and seniors made 7 A's, 6 B's, 15 C's, and 2 W's.

Of the sixty students, eight were given A's for the course on the Pre-Test, although they were required to meet in rap groups for three sessions and to present a performance program. Sixteen others finished a short time before the end of the term. Only four students did not receive credit for the course.

In an effort to validate the testing procedures and instructional effectiveness, Dr. Richard Cave analyzed the test results and studied a control group to determine the effectiveness of the class. His report follows:

I. Test Reliability.

The sections of the Pre-Test and Achievement Test dealing with characterization and structure were used in this evaluation. (The part of the test dealing with meaning only reflected students' perception of ideas and was not measurable in the usual sense.) The instructor scored the tests for the purpose of assigning grades to the students in the class. His scoring procedure was communicated to two control graders, and all scores were intercorrelated in a multi-trait-multi-score matrix. The matrix consisted of the Part I and Part II scores, pre and post for each grader: a twelve by twelve matrix.

The average correlation among the graders for Part I of the Pre-Test was .680; for Part II, .726. For the Achievement test the correlations were: Part I, .756; Part II, .685. All were around .7. The inter-scorer reliability was, therefore, more than adequate to conclude that the internal consistency was sufficient to use the tests to evaluate the effectiveness of the class. The number of students for whom we had complete data for all three graders was 56, and so 56 is the N used for the reliability correlations.

II. Effectiveness of the class.

Two groups were given the pre and post tests. The experimental group was the class of summer 1971. The control group was the class of Fall 1971, and this group took both tests (pre and Achievement) at the beginning of the trimester. The experimental group took the pre-test at the beginning of the trimester, and the Achievement test at the end of the trimester as a final exam.

Direct comparisons of the pre and post data for experimental group showed that the students improved an average of 7.72 points on Part I, but went down 3.98 points on Part II, as graded by an instructor. ($N = 57$; standard deviation of the difference scores for Part I was 11.35, and for Part II 11.24.) Using students to test for significance, the t for Part I was 5.14 ($p > .001$) and for Part II t was -2.67 ($p > .01$). The differences in the means could be explained on the basis of difficulty. If Part II of the pre test is much more difficult than Part II of the post test, the difficulty would more than compensate for the gains made by the students during the class. By comparing the data from the control group it appears that the pre test is, in fact, more difficult than the post test. Table I gives the means for each group.

TABLE I

Means and differences between means for the pre and post tests

	X_1 = Pre		X_2 = Achievement		$X_1 - X_2$	
	Part I	Part II	Part I	Part II	Part I	Part II
experimental group	28.45	42.55	36.30	38.45	7.85	-4.10
control group	29.51	41.61	18.72	21.12	-10.79	-20.49

The experimental and control groups do not differ by any substantial amount, but the experimental group means on the post test are almost double those of the control group. Assuming that the control group means show the relative difficulty level of the pre and achievement tests, (this assumption could be invalid if the differences between control and experimental scores resulted from the experimental students' being more motivated because they took the test as a final exam) then the real measure of the gains made by the students who took the class can be found by taking the $X_1 - X_2$ entries from Table I and subtracting the control valves from the experimental. So the gain made on Part I was 7.85 -(-10.79) or 18.64. The gain on Part II was -4.10 -(-20.49) or 16.39. We would conclude from this that the data indicates that the students who took the course made substantial improvement in their scores.

CONCLUSION

Findings. Developing and testing the models in mathematics and literature shows that the systems approach is workable and even desirable in many ways. The principal findings are described below.

1. The instructional design which begins with behavioral objectives and establishes learning paths leading to those objectives does, in fact, help students attain those objectives. The testing processes for the two models show students made significant gains through the experiences and instruction of the courses. Students in the mathematics model made definite improvement even after scores were adjusted for chance correct responses. With the literature model, where a control group was used, the learning effectiveness was clearly substantiated.

2. Although the development and formulative revision of learning materials are very costly, averaging nearly $10,000 for each of the models, the actual cost per student credit hour to offer the courses is substantially lower than normal. Offering the mathematics model costs $14.47 per student credit hour, including instructor's salary, student help, material, and equipment/facilities costs. CASC shows an average direct cost per credit hour of $24.84 for all departments in seven representative colleges. For two of the seven colleges, costs were itemized by department, and mathematics costs were $53.23 and $27.87. Offering the literature model cost $9.31 per student credit hour in direct costs. The CASC study of departmental costs reported $24.01 and $23.50 for English at the two schools whose costs were itemized. The savings, then, make this approach not only feasible economically but actually extremely desirable.

3. Student response to an instructional design which took into account their competencies was altogether favorable. Of course, the possibility of testing out of a course was attractive to students. Yet very few students made an effort to accelerate their progress and to take the tests early. On the other hand, deficient students showed little resentment at having to do more work, especially when they were certain what they needed to do.

4. Students enthusiastically accepted the design's allowing them to work at their own rate. However, their response indicated a lack of motivation or incentive. In mathematics only 43% of the students in the remedial sections completed the course successfully; whereas 90% of those attending regular sessions completed the course. Achievement tests results in literature show that students probably did not master parts of the basic

instructional materials. Such a design, then, needs to give special attention to motivation, especially for poorly prepared students.

5. The rap sessions and performance programs of the literature model give encouraging evidence of student ingenuity and perceptiveness. Attendance at the rap session was better than 90%. In conversations with advanced student-observers, class members frequently said that they were more aware of their prejudices, their own part in conflicts with parents and society, and certain aspects of their own nature. Interaction with others can certainly help attain some of the goals of the humanities.

6. Students attending the mathematics lectures responded favorably to the instruction and performed well on tests. The response seems to indicate that the average and better than average student benefited most from the design. The practical quality of the experiences in the regular program's instruction seems also to have been a factor in student response.

7. The basic system of the models creates an almost ideal use of personnel. The instructor works closely with students in defining goals and establishing their rationale. He also extends his influence by working with advanced students as assistants. They, in turn, understand more clearly the goals and methods of their discipline. They also master concepts in the process of helping others learn. Students in the class should benefit from the system's relying on their incentive and effort.

8. The systems approach can be applied with measurable success to learning problems which are cognitive and to those which are affective. Although progress toward cognitive objectives is easily measured, as this study demonstrates, affective objectives are more difficult to measure. From the experience with the literature model, the research team is convinced that attitude studies, personal initial statements, and tests of responses to certain situations can provide measures for affective goals.

9. Clearly the plan to accommodate beginning general education course to the individual differences of beginning college freshmen is feasible. Although the plan involves considerable financial investment in the developmental stage, the systems approach is economically efficient to the point of more than covering the initial investment. More importantly, student response demonstrates an increased level of learning and positive attitude toward the instruction and the course designs.

Recommendations.

1. The systems approach should be applied to the increasing problem of an expanding college population, of which a growing

portion comes from minority groups, disadvantaged families, inadequate secondary education systems, and older adult groups which have no high school experience. In particular, Oklahoma Christian College should continue its experiments, refining these models and developing new ones.

2. General research with the systems approach should give special attention to affective goals. The academic community is still generally suspicious of the systems approach, and that is owing in part to inadequate descriptions of the affective realm and in part to the lack of systems models which illustrate learning beyond the cognitive level.

3. Future research should study the long term effects of the systems approach. It is highly desirable to see if learning programs affect the quantity and duration of memory retention. It is also desirable to see if such programs have any effects on student curiosity, desire to learn, or resourcefulness in meeting complex problems.

4. The motivational problems of the systems approach seem to be related to students' inability to adjust to working independently without direct supervision, frequent teacher contact, and pressure from examinations and grades. If possible, a control group should be taken through all their college experiences by means of systems designs to see what patterns of personal discipline and motivation will develop.

The research team, judging its findings from the development and experimental use of the models in mathematics and literature, has concluded that the systems approach does, in fact, offer workable solutions to the problems of different preparation and background among beginning college freshmen. The approach allows students to work at their own rate, evaluates students knowledge and skills, provides necessary instruction in accessible form, and accomplishes all this with economic efficiency. The plan tested in this research is not a universal panacea, but it demonstrates one way higher education can go in solving the problems created by a growing and educationally diversified college population.

REFERENCES

Ballit, Conrad A. On The Teaching of Literature. Champaign, Illinois: National Council of Teachers of English, 1964. (ED 033 973)

Belasco, Elizabeth Simendinger. "Science Education in the United States." Educational Technology, January 1970, pp. 19-21.

Booth, Wayne C., ed. The Knowledge Most Worth Having. Chicago: The University of Chicago Press, 1967.

Booth, Wayne, The Rhetoric of Fiction. Chicago: The University of Chicago Press, 1961.

Bozak, Irene M. "A Summer Project for Underachieving Freshmen." Improving College and University Teaching, XVII (Summer 1969), 208-211.

Braddock, Richard. English Composition. New York: The Macmillan Company, 1969.

Bry, Ilse, and Afflerbach, Lois. "Links Between the Humanities and the Literature of the Human Sciences." Wilson Library Bulletin, (January 1968), 511.

Collagan, Robert B. "The Construction and Evaluation of a Programmed Course in Mathematics Necessary for Success in Collegiate Physical Science." Journal of Research in Science Teaching, VI (1969), 358-365.

Connelly, F. Michael. "Philosophy of Science and the Science Curriculum." Journal of Research in Science Teaching, VI (1969), 108-113.

Cooley, W. W., and Klopfer, L. E. Test on Understanding Science. Princeton, N.J.: Educational Testing Service, 1961.

Cooperative General Science Project: Progress Report 1966-1968. Atlanta, Georgia: Cooperative Science Project, 1968. (ED 030-577)

Design for Progress Through the Seventies, Oklahoma Christian College, 1970.

Dixon, John, Growth Through English. Reading: National Association for Teaching English, 1967.

Drake, Raymond W.J. "Review of the Literature for Freshmen Orientation Practices in the U.S. American College Personnel Association and Colorado State University, Ft. Collins, 1967. (ED 030 920)

Duffy, Brady R., "Individualizing Mathematics." Audiovisual Instruction, XIV, (1969), 55-56.

Eliss, Albert F., and Harbeck, Mary Blatt. Behavioral Objectives in the Affective Domain. Washington: NSSA, 1969.

Eraut, Michael, Fundamentals of Arithmetic. (McGraw-Hill Basic Skills System). New York: McGraw-Hill, 1970.

Fenton, Edwin, and others. A High School Social Studies Curriculum for Able Students: An Audio-Visual Component to a High School Social Studies Curriculum for Able Students. Carnegie-Mellon University, 1969. (ED 030 672).

Fromm, Erich, The Forgotten Language, New York: Grove Press, 1957.

Genkins, Elaine Kivy. "A Case for Flexibility in Classroom Instruction." The Mathematics Teacher, LXIII (April 1970), 298-300.

Godshalk, Fred I., Swinefore, Frances, and Coffman, William E. The Measurement of Writing Ability. New York: College Entrance Examination Board, 1966.

Green, Mark. "Aesthetics, Criticism, and the Work of Literary Art." College English, XXX (October 1968), 60-66.

Hays, Gary D., and Haywood, C. Robert. "Liberal Education at Southwestern" An Interdesciplinary Approach.: Liberal Education, December 1967, pp. 526-539.

Hoffman, Frederick J., Freudianism and the Literary Mind, Baton Rouge: Louisiana State University, 1957.

Holland, Norman, The Dynamics of Literary Response, New York: Oxford University Press, 1968.

Jewitt, John W., and others. A course in Basic Mathematics for Colleges. Berkeley, California: CUPM, 1971.

Jobe, John, A Large Group Teaching Plan for Mathematics. Unpub. report, Oklahoma State University: 1970.

Kruglak, Haym. "High School Mathematics Background of College Freshmen before and after Sputnik." The Mathematics Teacher, LXIII (April, 1970). 339-341.

Lange, Crystal M., Auto-Tutorial and Mobile-Tutorial Laboratory Techniques in Nursing. Government Documents, 1966 (ED 014 960)

Lesser, Simon O., Literature and the Unconscious, New York: Vintage Books, 1957.

Luckin, William. "English Literature Not the Whole Story." The Times Literary Supplement, May 10, 1968.

Machetanz, Frederick A., and Others, A Pilot Study Evaluating the Use of Programmed Learning in the Teaching of Exponents to an Intermediate Algebra Class at Los Angeles Valley College, Spring 1967. Education Documents, 1967. (ED 014 971)

Maher, James E. "Economics, Good and Bad." Clearing House, XLIV (September 1969), 16-17.

Mandel, Barrett, John, Literature and the English Department. Champaign, Illinois: NCTE, 1970.

McMillan, Robert D. and Brown, Edward K., "Audio-Tapes as an Instructional Device in College Mathematics." J. Research in Science Teaching, In-press. December, 1971.

Moffett, James, A Student-Centered Language Arts Curriculum. Boston: Houghton Mifflin, 1968.

Moon, Robert G. and Davis, Robert D., Elementary Algebra, Columbus: Charles E. Merrill Publishing Co., 1970.

Morgan, Lewis, B. "The Calculated Risk." College and University, XLIII (Winter 1968), 203-206.

Morrow, Thomas J., Programmed Mathematics; Des Moines High School. Government Documents, 1965. (ED 020 064)

Muller, Herbert I., The Uses of English. New York: Holt, Rinehart & Winston.

Noddings, Nellie L. "Providing for Individual Rates of Learning in Mathematics." The Mathematics Teacher, LXII (November 1969), 543-545.

Pearcy, J.F.F., and Lewis, K. Experiments in Mathematics. Boston: Houghton Mifflin, Company, 1966.

Pooley, Robert C. "Curriculum Research and Development in English." Journal of Experimental Education, XXXVII (Fall 1968), 56-64.

Raygor, Alton, and Wallace, James, Guide to McGraw-Hill Basic Skills System. New York: McGraw-Hill, Inc., 1970.

Rokeach, Milton; and Others, The Nature of Analysis and Synthesis and Some Conditions in the Classroom which Facilitate or Retard these Cognitive Processes. Government Documents. 1965. (ED 002 958).

Saunders, Dorothy. "Composition in the Elementary School." Report of School and College Conference on English (New York: Barnard College, 1965), pp. 17-28.

Shumaker, Wayne, Literature and the Irrational: A Study in Anthropological Background, Englewood Cliffs, N.J.: Prentice-Hall, 1960.

Simonson, Solomon S. The Teaching of Composition. Champaign, Illinois: National Council of Teachers of English, 1966. (ED 032 347).

Sobel, Max A. Teaching General Mathematics. Englewood Cliffs, N.J.: Prentice Hall, 1967.

Squire, James R., And Applebee, Roger K.,High School English Instruction Today: The National Study of High School English Programs. New York: Appleton-Century-Crofts, 1968.

Squire, James R.,English Literature. New York: The Macmillan Company, 1969.

Steinmann, Martin. "Rhetorical Research." College English, XXVII (January 1966), 278-285.

Wheelwright, Philip Ellis, Metaphor and Reality, Bloomington: Indiana University Press, 1967.

Woodby, Lauren G. The Low Achiever in Mathematics. Washington, D. C.: U. S. Office of Education, 1965.

Ziegelmaier, Gregory. "A Course in Humanity." Improving College and University Teaching, XXXI (Summer 1967), 158-159.

APPENDIX A

Author________________________________Title_____________________

How long did it take you to read the piece?_______________________

What were your initial reactions (thoughts and feelings) to what was happening?

What sections gave you pleasure?

What sections puzzle you?

In a few sentences tell what the reading means to you.

Check the response which is the best general description of your reaction.

_____I liked the piece very much

_____I like the piece

_____I was unimpressed

_____I disliked the piece very much

_____I disliked the piece

APPENDIX B

Arthur Miller, *The Crucible*
James Wright, *Mutterings over the Crib of a Deaf Child*
James Baldwin, *Tell Me How Long the Train's Been Gone*
Lew Welch, *Chicago Poem*
Robert Marasco, *Child's Play*
Truman Capote, *A Christmas Memory*
Dennis Trudell, *Going to Pittsburgh*
Lawrence Ferlinghetti, *The Poet's Eye Obscenely Seeing*
Edward Albee, *The American Dream*
Marge Piercy, *The Morning Half-life Blues*
Alastair Reid, *Curiosity*
James Purdy, *Cutting Edge*
Robert Creeley, *The Immoral Proposition*
Philip Roth, *In Trouble*
Lawrence Ferlinghetti, *The World is a Beautiful Place*
Flannery O'Connor, *Everything that Rises Must Converge*
F. R. Scott, *Examiner*
Paul Zindel, *The Effect of Gamma Rays on Man-In-The-Moon Marigolds*
Laurence Joesphs, *Passover at Auschwitz*
Irwin Shaw, *Act of Faith*
Eugene McCarthy, *Three Bad Signs*
Shirley Jackson, *The Lottery*.
Albert Camus, *Caligula*
John Wain, *Reason for Not Writing Orthodox Nature Poetry*
E. B. White, *The Second Tree from the Corner*.

APPENDIX C

Literature Instructional Essays

Literature: The Context

Introduction: The Importance of Context.

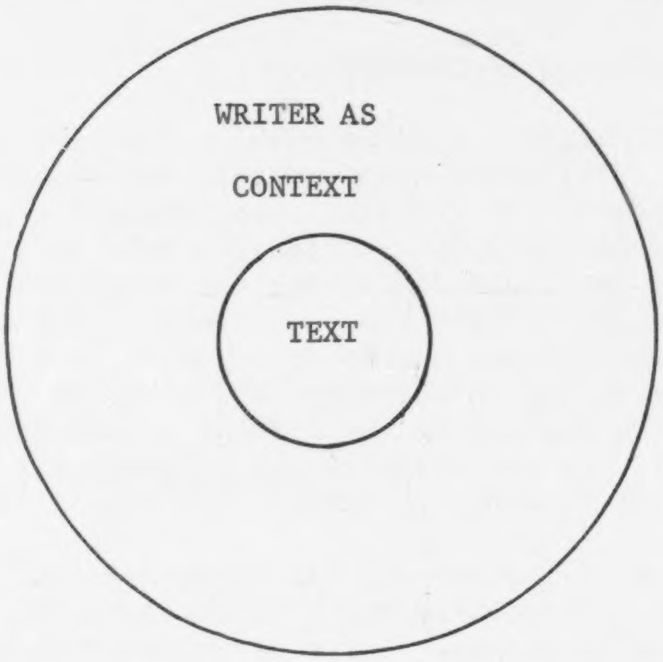
Literally context implies the relationship of a word or phrase to the surrounding text. Politicians complain that newsmen twist their statements by taking them out of context. Lady Macbeth's pathetic cries are sometimes wrested from their context as some joke to a dog--"Out, out--damned spot." And the lines describing the dangerous plight of the ancient mariner and his fellows has been appropriated by tipplers--"Water, water everywhere, but not a drop to drink!" In a broader sense, however, context refers to the surrounding circumstances or atmosphere. To illustrate simply, the diamond is the text of a work and the ring mounting is the context. Arthur Miller's The Crucible, is the text; all the circumstances surrounding its composition constitute the context.

Although one school of present-day criticism does not consider the context of any value in interpreting and understanding the text of literature, the context is an invaluable source of information and insight about meaning. One should not, however, misplace his values and study the context more than he studies the text. To borrow from Hamlet, "the Play's the thing" by which our feelings are caught and changed. The text, then, helps us more in our search for our own humanity--our capacity to think, feel, create, and communicate.

Context, on the other hand, may provide the information or insight about a text so that the text will be meaningful to us. Incidentally, it may well be that the greatest works are those which are read and enjoyed with no contextual information, but even those works usually gain meaning from increased background information. The context mainly helps clarify the circumstances and events which make up the piece. Context also provides vital information about philosophies and ideas in vogue at the time of the writing. Such information enables the reader to see a work's ideas in relation to philosophic, political, or economic thought. A third value of context is its helping the reader understand distinctive or particular uses of language. Finally, background information may confirm a reader's inferences about parallels or allegories which may be too subtly presented. Contexts add to understanding in many ways--some so indirectly that the reader is only vaguely conscious of seeing or understanding more. Background information is, therefore, important to the fullest possible understanding of a work.

I. The Writer as Context

Because text is actually the creation of the writer's mind, the chief factor in the context is the writer. The diagram of concentric circle is one illustrating the relation of text to autorical context.



For the sake of simple analysis, the writer's context may be considered in three categories: 1. Biography, 2. Historical era, and 3. Psychology. Obviously all three of these aspects are closely interrelated--almost to the point of being inseparable.

In a sense, biography consists of all those traceable, overt things a man does all his life. Most writers use in a transformed way, the experiences of their lives. Tennessee Williams, for example, creates The Glass Menagerie from his experience with his mother and his sister. Ernest Hemingway turned almost all the adventures and crisis of his life into the substance of his stories. His World War I ambulance driving days are reconstructed in A Farewell to Arms. His participation in the Spanish civil war gave him the material of For Whom the Bell Tolls. William Faulkner, on the other hand, passes over the events of his own experience as subject for his fiction. But he uses his home town, the countryside and whole Oxford, Mississippi atmosphere as the setting for his work. Henry James illustrates how the thoughts and philosophies a man encounters can become the substance of fiction. Almost everything William James, the novelist's famous brother, said about human perception and sensitivity is directly illustrated in Henry James' fictional world.

In a very special way the other writings of an author become a vital element in the context of the work. William James wrote prefaces to all his novels in order to explain his methods of creating fiction. When a group of critics gave Lord Byron's poetry bad reviews, he retaliated with a scathing satire on English Bards and Scotch Reviewers. Some writers--notably William Faulkner and Honore de Balzac--use the same characters from novel to novel, and each succeeding story clarifies the character's true nature. Salinger's Glass family saga is the best contemporary example of continuing set of characters. Luigi Pirandello illustrates another way writer's other works provide context. His play--Six Characters in Search of an Author--aroused so much confusion and conjecture that he felt compelled to write It Is So, (If you Think It Is) to clarify his views about the relativity of truth. Other writers keep diaries and journals about their work and their plans. The journals of Nathaniel Hawthorne reveal much about his novels. Thus, the other writings of an author and biographical detail constitute vital parts of context.

The text a writer produces will be strongly influenced by the era in which he lives. One of the most immediate effects is on the form a writer selects. If Shakespeare had lived in the nineteenth century, he might be famous for novels instead of plays. Hawthorne's parabolic tales could never have been written if he had lived in the 20th century. Presently a lot of aspiring writers--even those at colleges and universities, are writing poetry, which is enjoying a new popularity. Ten years ago, poetry seemed a dead art, and all the students were talking of writing the great American novel.

An author's age will influence him to write of the great events he witnessed. Often the great events of an age so affect the writer that he incessantly treats of one or two of those effects. Charles Dickens, having risen from the poverty and deprivation, constantly tells of young boys trying to rise to success in middle class London. Leo Tolstoy's War and Peace reflects his age's recollection of Russia's greatest hour--repulsion of Napoleon. Without the great depression and Oklahoma's dust bowl, Steinbeck could never have written The Grapes of Wrath. And F. Scott Fitzgerald's works are permeated by the Jazz Age hullabaloo of the 1920's. The events, then of a writer's era become an important part of the context surrounding his works.

The ideas current in a writer's time will profoundly influence a writer and will, therefore, be important aspects of context. Everyone knows that a knowledge of transcendentalism illuminates many difficult parts of Whitman's Leaves of Grass. Similarly, Shakespeare's plays mirror the great chain of being concept which ordered the social and political life of the Renaissance. Some present writers reject the immortality of man and the concept of an absolute being. Consequently, their works depict man in a hopelessly absurd world with no values, no achievements. In the highly successful movie, Beckett, Anouilh has Sir Thomas a Beckett go to meet his assassins--in an act conventionally viewed as courageous and godly--with the words, "Now, the great folly begins." Current ideas, then, exercise a subtle yet powerful influence on a work, and consequently, become a vital component of the work's context.

The historical era as a context for a work consists not only of its current views of life but also of its encouragement or discouragement of artistic effort and freedom, its tides of social movement and change, its feeling for men or institutions, and its general spirit toward life and mankind.

Obviously, the personal experiences of life and the historical background before which that life moves have a vast and almost indeterminable effect on the psychology of the writer. But that psychology reveals itself in every stage of the creative process. From the original fantasy with which the writer begins through developing of episodes, description of scene and atmosphere, organization of conflicts, definition of character--even word choices and combinations--the writer's subconscious nature affects his choices. This intricate process is, however, still virtually beyond man's understanding. A few psychiatrists have tried to reconstruct parts of the psychic for seemingly disturbed writers like Jonathan Swift and Edgar Allen Poe. Psychologists have tried to show some relation to writers and their work. Newton Arvin, for example, has drawn many interesting conclusions about the sexuality of Herman Melville from a psychological reading of *Moby Dick*. Although these studies are meant to be tentative and tend merely to psychoanalyze the writer in a superficial way, it is important to recognize these principles.

1. The *fantasy* of a writer's poem, play, or story springs from the *subconscious* and is closely related to *dreams* as a psychological function.

2. Although writing is a *conscious*, *creative* effort, it is affected constantly and powerfully by subtle *subconscious* attitudes.

3. Usually a man's writing will reveal many qualities about his *personality*, but one should be cautious in dogmatic judgment, for distinguishing what the writer does consciously and what *unconsciously* is impossible.

4. Fantasies which are latently *sexual* or *amorous* are those which are most easily and universally *perceived*.

5. Fantasies of *fears* and *aggression* are also easily *discerned*, but usually have less appeal to readers.

6. Fantasies are generally parallel to the *ancient myths*, which some anthrologists now see as persistent expression of common *human fantasy*.

The writer's psychological nature as the context of a work is certainly harder to recognize and detail than are the geography and historical era, but that psychology is powerful and more intriguing. Although readers should be leary of becoming amateur psychologist, they can speculate about the psychic nature which activated the text of an interesting piece.

II The Reader as Context

As important as the writer is as context, he is only half the context. The reader establishes a vital context for any story. The reader approaches every text with his own biography as the sounding board against which he tests the events, characters, and emotions of the fictive world. A reader often responds favorably to circumstances he understands well. For example, when I read Bernard Malamud's A New Life, I stayed with the book because it told of a college English teacher's misadventures in Acadimia. Surrounded by a kind of formalism I have seen at many universities, the hero, fights valiantly for putting ideas back into English. I'm sure I read into that story many of my own experiences. Similarly, a student of mine once became very hostile about reading Lord of the Flies. Talking with him, I discovered he had once been fat like Piggy and he consequently, understood too well the jibes and insults with which the boys bombarded Piggy. Our own biography becomes a living context of any work we read.

Then too, we bring our own times to form part of the context of any works we read. Living in the shadow of the atomic mushroom-shaped cloud, most readers of Albert Camus' The Plague fully comprehend the sense of uncertainty, the hysteria, and the patience which mark the human attitudes toward death. Our present wars tend to distort our perception of fictive wars we read about. Of course, our age influences us to read certain books. The school system may coerce us into reading Shakespeare or Dante. Social pressure may drive us to sludge through Love Story. Consequently, what we read and how we read it will be greatly affected by our own times.

The reader's context includes, as a third quality, his own psychological nature, which ultimately accounts for all his likes and dislikes of what he reads. All three of these factors qualify the perceptions of the text in powerful ways. The reader may improve his perceptive levels if he remembers these principles.

1. In an intense state of excitement about his life, the world situation, or his own ideas, a reader will not easily enter the fictional world of his reading.

2. Although a reader may avoid circumstances and occurrences in life which arouse anxiety or fear, he may be able to cope with those problems in an imaginary world.

3. A reader may not be able to see clearly characters or incidents in a story if he has trouble recognizing motivation or judging actions in real life.

4. A reader's experiences with fiction has the same kind of effect on his life and thought that his real experiences have if he has read wisely.

5. A completely well-adjusted person will not read fiction because he has no need for the psychic release of fantasy.

6. A very poorly adjusted person will not read fiction because the fantasy arouses fears and aggressions which he will not be able to handle.

In conclusion, knowing about the reader as the context may help you read more effectively because it may allow you to not waste time trying to read if you are disturbed or upset. It may help you understand some of your inexplicable likes and dislikes. But most important, it should make you trust and rely on your responses to literature. If you and I see Macbeth in different ways, it is because each of us has read the play in the context of his own experiences. What we have seen has been colored by our vastly different lives.

Tips for Rapping the Short Story

Your job as a student-critic is to learn how to describe--in papers, in class discussions, in rap sessions, and in talks with your friends--the art of short stories. This essay suggests some questions you can ask and some terms to help make your descriptions orderly and illuminating.

To study the art of something means to examine how it is put together and why it is put together in that way. Such study requires us to determine the purpose of the whole so that we can then explain how the various parts function or contribute towards that purpose.

But what is the purpose of a short story? A story is intended for a reader. He reads it and contemplates it. This contemplation is a kind of experience, and the value of the work of art lies in the vividness and power and interest of the experience for the reader. We describe what we see when we look at the story, and in this way we begin to understand what its art is. The aim of a short story is to present to our imaginations, by using the resources of prose language only, an image of things happening in a fictional world. Your analysis of the art of the story will be most efficient when it goes to these problems first. (1) How has the author met the task of inventing fictional events in a fictional world capable of arousing the strong interest of the reader? (2) How has he met the task of rendering or displaying these fictional events vividly enough to be impressive?

The challenge to you is to intellectualize, to put into words what you see and feel in a story. This is not easy. Short stories deal with feeling, and feeling is always difficult to describe or analyze. Do not be afraid to try, for what a good analysis does, after all, is bring things into clearer consciousness. It focuses the mind. How to discuss the short story is the problem we invite you to consider.

A story is about people. These people exist in our imaginations, but a good writer will always help us by making the imagined person more vivid and real. This he does by giving him what we call character. We must distinguish between a character, referring to a person in a story, and character, referring to the qualities that make him the kind of person he is. A character has character.

Character may include any trait that belongs to a person in a story although the term is usually limited to those which cause him to act, think, and feel as he does. Character also includes attitudes--likes and dislikes or ideas of good and bad that cause a character to behave as he does.

Character in a story may also include a moral evaluation of the person. He may be presented to us as good, bad, or ordinary. The job of analyzing a character includes the task of judging the character. We can usually detect an attitude we are expected to take towards a character, and we must look for signs of that attitude and try to respond to it.

The most important way in which character displays itself in a story is through the choices and character makes. A choice can be positive or negative, a refusal to act as well as an act that changes things. Character also manifests itself in the feelings with which a person responds to a situation, for the feelings reflect attitudes and thus indicate potential choices. Character may also be indicated or suggested in other ways. The aughor himself may tell us about a character and his traits; other characters may note them; a character may describe himself. Character may also be suggested quite indirectly, by associating a person with certain images or symbols or by connecting him with certain stereotyped notions of personality ghrough physical appearance or occupation.

Character can change in a short story. A stage in somebody's growth may be depicted. An immature attitude may be replaced by a mature one. In many stories, the interest depends almost entirely upon how the characters reach an understanding of what they think and feel about what happens to them. It is their thoughts and feelings which make the events meaningful to us.

All short stories do not have a plot such as entanglement, rising action, climax, and denouement; all stories do have conflict. A conflict simply means that a story brings together two opposing forces which we call a protagonist (that is, "one who struggles for") and antagonist ("one who struggles against") and then resolves the resultant struggle between these two forces. We can classify many stories as accomplishment or decision stories. In the accomplishment story the main character tries against opposition to achieve some goal; in the decision story he is forced to choose between two things--two sets of values or two courses of action. Some stories do not fit into either of these categories. Conflict is a necessary element in the short story; plot is not. Conflict in a story must be significant; it must be of obvious importance to the characters involved. We call fictional conflicts crisis situations, meaning that as a result of a given conflict, the character or characters involved will never again be quite the same people that they were before the incident occurred.

One characteristic of conflict in the short story is that some kind of change is effected in the character involved; another is that the two forces which constitute the conflict must be reasonably equal in strength. The skillful writer will balance his opposing forces so that his outcome remains in suspense until the end of the story. A conflict must be plausible. Plausibility is that inexplicable quality of a story which makes us believe. The most convincing story is the one that most closely approximates life as the reader has experienced or observed it.

A story must have unity. This means everything in a story--the conflict, the characters, the theme, the symbols, the point of view, the incidental devices--must be interrelated, functional and necessary to achieving the story's basic purpose or effect. Each development in the conflict of a story must follow logically--if possible, inevitably--from a preceding development.

Significance, unity, and plausibility are the necessary characteristics of effective fictional conflict.

The writer lives in and observes human society. He comes to generalizations about human beings and their problems. One way he transmits his generalizations to others is by telling a story. This generalization, stated or implied, that lies behind the narration of a specific situation involving specific individuals is what we call theme. Theme exists in fiction because human beings live in the same world, share similar emotions, react in similar ways to similar situations, and face common problems. When a writer comments on his particular, immediate world in such a way that his observations apply to no other time or place, we say that his theme is topical. On the other hand, when a writer deals with problems and subjects that concern all men, then we say his theme is universal. The matter of theme in fiction and the distinction between topicality and universality are important for they frequently mark the distinction between the significant and the insignificant in art.

The idea of a theme should not be equated with that of a moral. Serious short stories deal at times with ethical or moral values, but they are often concerned with challenging, nor reaffirming, moral platitudes. Good stories do not teach; they reveal: they do not preach; they interpret.

Theme is usually handled in fiction by implication. The serious writer wants to convince his reader of the validity of his interpretation of experience so he appeals emotionally to the reader by subjecting him to an experience that will force him to feel that same truth.

Most of us can follow what happens in a story; most of us can also see what a story's characters are like. But implied theme is another matter. The first step is to determine what the story is about. Every thematic story concerns itself with some subject--materialism or individuality or justice or death or ethical values, but the subject of a story is not its theme; the theme is the statement the story makes about the subject. To determine what that statement is, look at the characters in the story. For a story to have meaning, the people in it must have character traits, or attitudes, or concepts, or something in common either with the reader or with people in the reader's world. Consider in what ways the characters in the story are typical or representative. Do the same with situation. The situation in which the characters find themselves and the problems they face will ordinarily parallel in some respects situations in which we have found ourselves and problems we have experienced or observed. Then, with character and situation thus analyzed, consider how the characters react to this situation and what happens to them as a result. From this analysis of a specific situation, it will be a short step to a generalization which, in effect, will state that if we have this kind of a person in this kind of a situation, and if he reacts thus, then this is what will happen to him. This is the theme.

The most effective stories are those which interpret honestly and intelligently a genuine human problem. They interpret life emotionally by subjecting us vicariously to the ordered experience.

Structure in the Short Story

Structure is the mechanical form of the story. It is the shape, the skeletal rigidity, or design of the story. Each story's shape evolves in relation to its material. That shape always involves movement. For example, a story may take a circular shape. On its ascending curve, it might portray a voyage to a certain place. When the circle reaches its highest point, an incident happens and as a consequence the voyage is repeated as a downward movement to the point of origin.

A common structure has been suggested by the shape of an hourglass laid on its side. This kind of story usually involves two contrasting characters or two opposing attitudes. When it is characters, one character is represented in poor circumstances at the bottom of the glass; the other is represented in excellent circumstances at the top of the glass. As the story progresses, the movement sends the character in poor circumstances from his position at the bottom of the glass to a corresponding but opposite position at the top. Meanwhile, the character in excellent circumstances is making a descending movement towards the bottom of the glass. This type of structure can be compounded into a double hourglass. The roles are reversed not once but twice and the story ends where it begins but with certain knowledge added.

Some structures may involve background legends defining the shape. The structure may become mosaiclike in its intricacy of pattern.

Structure is the most detachable element of the short story, yet it is difficult to talk about structure without talking about theme. For example, the characters illustrating theme may move from one position to another, and they may express ideas related to the theme. Structure may be achieved by shifts in point-of-view, but the shifts also focus the reader's attention on theme. A pattern may emerge and guide the reader as he sees just where the story is going. Much of this pattern would be thematic.

Structures may, on the other hand, have little effect on theme but much on the condition of a character. Structure may also be composed of blocks of action moving towards a final confrontation. A story may have a double structure of past and present, like a double-exposed negative, where the details of each are sharp at certain points, but the whole is purposely blurred by the imposition of one view over another. A short story may have a similar structure in its contrasts of periods in time. In such structure the clear, sharp periods are almost as important as the characters themselves.

The concept of structure as a beginning, a middle, and an end is helpful because it speaks of a necessary relation between the three parts, but it is limited to a logical ordering of parts. Another helpful concept is structure as an image of the human body where parts are necessary and proportioned.

Now let's take a closer look at the structure of an action in a short story. The structure of an action as presented in a short story is the plot of the story. It is not the structure of an action as we happen to find it out in the world, but the structure within a story. The plot is what happens in a story. It is the string of events thought of as different from the persons involved in the events and different from the meaning of the events. In other words, the structure of an action within a story is what the writer has done to the action in order to present it to us.

When we talk about an action, we are not talking about a single event. We are talking about a series of events, a movement through time, exhibiting unity and significance. It is a series of connected events moving through three logical stages--the beginning, the middle, and the end.

The beginning of an action always presents us with a situation in which there is some element of instability, some conflict; in the middle of an action there is a period of readjustment of forces in the process of seeking a new kind of stability; in the end of an action, some point of stability is reached, the forces that have been brought into play have been resolved. In an action, there must be unity and significance. That is, it must move toward an end, and the end must settle something. The unity and significance of an action must be inherent in the action. In other words, the patterns giving unity and significance of an action must be grounded in the facts.

When we talk about facts in a short story, we are not talking about historical facts or actual happenings. We are talking about imaginary facts which the writer places in some special perspective of interest and interpretation so as to constitute an action. He manipulates the action in order to turn it into a plot. He selects the facts that are useful for his particular purpose. A fact is useful if it is vivid and stirs the imagination of the reader to accept the story. One vivid fact can make a whole passage of narrative seem real to the reader. A fact is also useful if it indicates, directly or indirectly, the line of development a story is pursuing, that is, if it indicates how one thing leads to another, or what is the meaning of the movements of events.

Facts in a short story are not always ordered in the strict chronological sequence of an action. For example, the beginning of a story may, or may not, pick up the beginning of the action which it presents. The plot may plunge the reader into the very middle of things, and then, stage by stage, take him back to the beginning of the action. There may be a complicated intervening of times. So we see then that the plot is the structure of the action as presented in a piece of fiction.

When we talk about a story, we are talking about the idea of a unity. We are implying that the parts, the various individual events, hang together. This brings us to the matter of cause and effect. In any story, we expect to find one thing bringing on another. If we can detect no reasonable connection between them, if there is no logic whatever, we lose interest. Every story must indicate some basis for the relation among its parts, for the story itself is a particular writer's way of saying how sense can be made of human experience.

Cause and effect is one way of saying that sense can be made of human experience. When we talk about cause in the short story, we are talking about cause in human events and human responses to changing situations, including the possibility of action taken to change existing situations. Many non-human things may enter into the logic of a story, but the central logic for the reader is the logic of human motivation. So we might think of a plot as being character in action.

Logic, including the logic of motivation, binds the events of a plot together into a unity. This unity involves change. At the beginning of the story we find a situation with some element of instability in it. At the end of the story, we find that things have become stable once more. As we have said, something has been settled. This movement from instability toward stability involves certain natural stages. These stages are the beginning, the middle, and the end.

The beginning of a plot action is called the exposition, i.e., the circumstances from which the story will develop. The middle is called the complication, for it presents the increasing difficulty encountered in the movement toward stability. The complication moves toward a moment, an event, when something has to happen, when something has to crack. This moment is the point of highest tension, the moment when the story turns toward its solution, the climax.

The end or denouement gives us the outcome of the conflict, the solution to the problem, the basis of a new stability. This basis of a new stability may be only provisional and temporary. No promise is made of an end to all conflict, all struggle, all problems. The denouement simply is the settling of a particular action, which is the story the author has elected to tell.

Structure with all its complexities is simply the form by which a writer creates characters, develops theme, and conveys meaning. Structure, at its best, helps the reader perceive the facts, implications, and meanings of a short story.

Structure in Drama

The lives of most people fluctuate between the extremes of tragedy and comedy. They travel between the two extremes on a journey that brings them into any number of conflicts wherein they lose. Instead of being destroyed, however, they adjust to the new circumstances of life and proceed toward their next conflict. As a result, most people live a constant drama. Some moments are more intense than others, more fraught with danger, suspense, and distress; these are the dramatic times. As people adjust from those brief encounters, they laugh a little at fate and at themselves; they think of the absurd elements of their situation, for laughter is man's salvation. Most people alternate so regularly between the dramatic elements of life and those elements which serve as comic relief that life becomes a tragicomedy.

Theatrical drama embracing the two extremes of tragedy and comedy with all the intervening gradations is an attempt to emulate the wide range of these possibilities of a man's life. The basic themes of drama are the themes that are constant in all literature whether in poetry or in prose, the short story, or the novel. They include considerations of love, hate, fear, greed, sexual need, spiritual fulfillment or any concern in the life of man.

The drama has always been patterned on man's concept of life. Man knows that his activities are bounded by time and take place within a framework of physical geography. That awareness creates the setting of his play--the background in time against which the action of his life will occur. So it is with the setting of a stage production. The geographical locale of a play sets the immediate atmosphere or mood, and the playwright's selection of characters, words, and general attitude supply the tone of the play as characters move and speak against that background. The tone may be humorous, pathetic, or ironic. The reader must share the attitude of tone if he is to understand the meaning of a line.

With the set furnishing the time and place of action, characters move through time and place. Each time a major character enters or leaves the set, a scene begins, for, as in life, brief incidents occur when people come into contact with a place or with other people. In isolation, any incident or scene in a play may have little meaning, but because it is affected by scenes which have occurred before it and it will affect scenes which come after it, the scene or incident forms a part of the total pattern that is the larger action.

A playwright carefully considers the importance of his scenes, assesses their value in making the total play clear to the reader, and chooses those scenes which contain the most intense dramatic possibilities. Scenes may serve any one of many functions. Expository scenes present explanations the reader needs. Messenger scenes including verbal messages, letters read aloud and telephone conversations introduce complicating matters in an easy fashion. Transition scenes bridge two important scenes that would otherwise seem unrelated. Development scenes reveal the impact of one complication on another as they are joined together. Climax scenes contain the point of highest interest of the larger unit. These scenes are then woven into larger units of action--units which reveal their meaning as the pattern of related scenes emerges.

The larger unit to which each scene makes its unique contribution is called an act. An act is basically analogous to the days, months, and years in a human life. An act is a measured space of time; the period between acts marked by an intermission, a falling curtain or some other device allows time to be abridged or passed over.

An act may consist of only one scene or several scenes. The playwright decides how best to involve the reader, present the problem, reveal complications of the problem, reach a climax which demands resolution, and polish off the work. This decision creates the number of time divisions in the play that make it a one, three, or five act play.

In the five act pattern each act fulfills a particular function leading to the success of the larger unit. For example, in a typical five act play, Act I serves as an introduction or exposition. It presents the setting, the characters, and the background action needed by the reader if he is to understand the present action. It also creates the atmosphere and tone of the play.

Act II in a typical five act play presents rising action or complication. This is the action the reader is concerned with once he has adequate background to hold his interest. For this purpose, the playwright ordinarily employs an exciting force. Actually, the exciting force usually closes Act I and paves the way for the reader's concern in Act II. The second act is devoted to creating a series of complicating acts or mental conflicts between the dominant characters of the play.

Action continues to rise in such plays until, usually in Act III, a turning point or crisis is reached. Up to this point, the main character has probably managed to succeed in whatever he is attempting without too much hindrance. Suddenly the turning point is reached and he overreaches his good fortune.

Falling action is the dominant influence, usually late in Act III or early in Act IV, of the forces in opposition to the central character. Suspense must be maintained at this point and any weakness of the play will manifest itself here. The term "falling action" should not suggest a decline of dramatic interest; in fact, interest intensifies at this time. The climax or the point of highest reader interest usually occurs in this act.

The castastrophe occupies Act V as the punishment or the death of the central character and/or opposing characters comes about. The action of the play comes to a logical conclusion, and the reader is s satisfied with the restored order of life.

Today, three acts is the accepted standard, but the basic plan of introduction, rising action, turning point, falling action, and catastrophe is retained. Whether in five or three act form, a play begins with exposition, continues on concentrated refinement, and closes on intensified response, anticipation, or outright resolution. At that time, the theme or basic overall statement of purpose of the play should either suggest itself or be offered in such a way that the reader can apprehend it.

Playwrights manage to reveal truth, to make it meaningful, by creating a set of circumstances, peopling those circumstances with characters, giving those characters carefully controlled speeches or dialogue, and bringing their created world to a conclusion that, in its powerful, directed climax, forces us to understand truth.

A one act play is a remarkably easy to read form because it shares so much in common with the short story. One act playwrights and short story authors alike aspire to brevity, a single incident illuminating a character's total life, one dominant effect, and a revealed change in the major character.

Notes on Poetry

A poet has different types of structure or modes of presentation from which to choose. By structure here we mean certain facts of the interior character of the poem. We do not mean its external shape as number of lines, length of lines, or arrangement of rhymes.

Just as we have the structures and shapes of cats, fish, monkeys, birds, lions, and men in the world of living creatures, so we have, in the world of poems, such structures and shapes as elegies, lyrics, epics, pastorals, and ballads. The poet may tell a story, describe a scene, present a feeling or idea, or advance an argument for an attitude or idea. These types of structure or modes of presentation we may label, respectively, narrative, descriptive, expository, and argumentative. In most poems two or more types of structure are present in combination, but in many poems one type dominates.

In the simplest narrative poem, the indispensable elements are the problem, the turning point, the climax, and the Denouement. In most such narrative poems, there will also be an element of exposition to give the background of the characters and the initial dramatic situation. The simple narrative structure of a poem is closely akin to the simple narrative structure of a short story--exposition, development, a turning point, and a clearly depicted climax.

Description--as a type of poetic structure--appears more frequently as a subordinate than as the predominant element. But there are some poems, usually rather brief, in which the descriptive element is primary. In such poems the poet is concerned with presenting a scene in words, with conveying all the sensory richness of his subject without depending upon the interest of event or character.

So we see that structure is the arrangement of materials within the poem. If there is a time sequence in the poem, then the structure might be described as chronological or narrative. Each poem has its own way of relating the parts to the whole poem. With each new poem the reader must discover the structure anew.

One of the most efficient means for laying bare the structure of a poem is to ask and answer this question: "What are its main parts, and what is the relationship of the parts to each other?" Many poems have two main parts. The parts may be contrasted with each other.

In a successful poem the inner structure harmonizes with the outer form. The interaction of the inner structure and the outer form is dramatically clear in a sonnet if the sonnet is a good one. For example, each quatrain of an English sonnet usually develops different aspects of a single thought. Or in an Italian sonnet, the first quatrain might present a series of visual, auditory, and tactile images; and the second quatrain might explain the significance of those images. Together the two quatrains would form the octave, which present the problem. The sestet would not solve the problem but would relate it to a larger problem. Thus the poem would move from particularity to generality. In an English sonnet each shift in rhyme scheme might signal a new development of the thought.

A convenient way to think of poetic structure is to think of it as a progression. Progression is the means by which a poem reveals itself--the means by which it moves from its beginning to its end. As we have said there are four main types of progression: Narrative, descriptive, argumentative, and expository. Every individual poem has a progression unique to itself. If a reader is to experience a whole poem, he must be aware of how and where it is moving.

Let us go back to the narrative progression. A poem whose details are organized, as we have said, chronologically has a narrative progression. Such a poem tells a story, and it may have, as has been stated, all or some of the features of plotted fiction: a rising action in which a conflict develops, a climax in which the conflict takes a decisive turn, and a falling action in which the conflict reaches the conclusion. So again, any poem whose details are arranged in a time sequence has a narrative progression. In other words, there is a sense of advance. What comes later in the poem could not come earlier. The poem seems to get somewhere, to settle down to an end. It is built on a series of incidents moving to a denouement, an outcome or resolution. With varying emphasis, the narrative pattern presents characters, settings, and movements in time. In the basic narrative pattern, the incidents are central.

In handling the narrative structure of a poem, the reader should identify the incidents or episodes. A guide to these would be a change of time, a change of setting, or a turn of events. He should identify the narrative pattern. For example, he should recognize the denouement, which is the product or outcome of the events that precede. He should clear up expository details by identifying settings and changes of settings, different speakers and characters, and time and time-movements. He should understand the cause-and-effect relationship between incidents. He should try to follow the progression of events from the beginning of the poem to the end. He should look for time words such as "when," "soon," or "till," which would signal incidents or turns of events.

It is impossible to talk about structure without talking about theme. So once the reader has identified the speaker or speakers and finds out who is talking or who is talking to whom and he studies the arrangement of the material, then he should be able to arrive at the generalization or the theme of the poem.

Let us go back now to descriptive progression. As was stated, a descriptive progression is an arrangement of pictorial details and there are a few purely descriptive poems. A descriptive progression relies on a rich sensory variety and enlists many different senses. Many facets of our complex visual sensation might be engaged. For example, present active participles might be used to make the whole description move. Punctuation might serve to indicate the continuity, the sequence, and the pace in time. The avoidance of a final period might imply the on-going of a process.

When a poet advances a proposition and then presents reasons in defense of it, he uses an argumentative progression. Few poems have only an argumentative progression because a person who merely wants to present an argument can do so more convincingly in prose than in verse. In logical or argumentative structure the speaker argues a case, and he comes to some sort of conclusion. In other words, the poet states a proposition and then gives reasons for the acceptance of the proposition or proof of its truthfulness. He may conclude with a restatement of the proposition.

Any arrangement by means of which a poet sets forth or exposes his ideas and feelings is an expository progression. Broadly defined, a progression of this kind must exist in every good poem.

Every poem raises and answers some sort of questions. One kind of expository progression is a movement from ignorance at the beginning of the poem to knowledge at the end. If a poem expresses a feeling, that feeling will be less vague at the end than at the beginning; if it expresses an idea, the idea will be clearer; if it describes a scene, the scene will be more vivid in the reader's mind.

There are many ways by which a poem may move from ignorance to knowledge, from vagueness to precision. A poem may do this by supporting a generalization with specific details. Contrast and comparison (any poem with an extended metaphor) are common expository devices. A succession of figurative comparisons or precise images help to make the subject of a poem stand out distinctly as itself and nothing else.

Another general kind of expository progression causes a poem to become more emotionally profound as it proceeds from beginning to end. This sort of progression is best studied in the lyric, any short poem expressing personal thoughts and feelings. In such a poem, as the reader becomes more mentally aware--that is, as his mind takes in what has happened--he becomes more emotionally involved. Since the material of lyric poetry is usually feeling or emotion, it is not surprising that the structure of many lyric poems is expository. The expository type of structure is also appropriate in poems devoted to the expression of ideas rather than feelings.

With many ideas, images, and metaphors packed into a very few words, the reader can be overwhelmed if he tries to grasp a poem all at once. He must first break the poem down to its smallest parts or minimum sense units; he can manage these. Once the reader has defined the parts, he can begin to see the structural pattern of the whole poem and he will have a definite framework for working further into the poem.

We have seen that the structure of a poem is usually built on one or more of four patterns--the narrative, the argumentative, the descriptive and the expository. The narrative pattern is a series of incidents moving to a denouement. The argumentative pattern is a progression of ideas moving to some kind of resolution. The expository pattern makes use of a dominant image or series of images to form a total impression and sets forth or exposes the poet's ideas and feelings. The descriptive pattern conveys all the sensory richness of the subject of the poem.

Let us go back to the argumentative pattern. In handling the structure of argumentation in a poem, the reader should identify the sentences and sense units. The sentence is the usual sense unit, but there are variations. The reader should look for compounds, qualifying clauses, groups of sentences as units, and the stanza as a unit. Punctuation marks indicate shifts. Colons, semicolons, commas, and dashes are helpful signs of different degrees of shift or interruption in movement of ideas.

The reader should also identify connecting words. Words like if, but, and thus, are tangible signs of an argument pattern. They are not only signs of separate ideas but they are also signs of a special kind of meaning relationship between the ideas. The reader should underline such connecting words; he should identify the ideas they connect, and he should study the relationship.

Connectives are extremely important in the argument pattern of a poem. For example, the connectives _or_ and _whether--or_ would mean an alternative; _and_ would mean a series or additives; _for_, a causal meaning; _when_, a temporal or conditional meaning; _but that_, an eliminative meaning; _rather_ and _than_, a comparative meaning; and _thus_, a result. There are many other common connectives. Some of the more common are _if_ meaning conditional, _although_ meaning concessional, _but_, meaning contrasting, _by_ meaning instrumental, and _in order_ meaning purpose. A reader should learn to identify these words as guides to meaning in an argument pattern.

In addition, the reader should also identify special patterns of words and phrases. There are standard verbal patterns that give definite signs of structure. Some of them are restatements, compounds, repetitions, rhetorical questions, series, interruptions, appositives, and synonyms.

The reader should identify the overall argument pattern. Sometimes making a paraphrase of the argument content of the poem is helpful. The close of the poem is a good place to start because the final sense unit often tells what the whole poem is about. The reader can see how the poem ends and then look back to see by what stages the poet arrived at his conclusion. The sentences, the connectives, and the special patterns will serve as guides. If the poem is a finished art form, it will have a resolution. The reader should keep in mind that the basic principles of analyzing argument structure dictate first a quick check of ordinary sentence structure and punctuation.

Let's take another look at the structural pattern of organization that we have called the expository pattern. Identifying the image or expository pattern in a poem may serve as an additional dimension to anlayzing the poem, or the expository pattern may serve as the most immediate way into first-level meaning. The additional dimension of the image or expository pattern expands our understanding. There may be images of action opposed to images of passivity.

In handling the image or expository pattern, the student should identify the images and then identify the image pattern. He should decide whether the images are parts of a single larger image or whether the images separate images joined by a single theme. He should discover the way the images support the structural pattern. For example, the structural pattern may be entirely one of images or the image pattern may be joined with another pattern, such as argument pattern or narrative pattern. The student should consider what theme, idea, or emotion the images suggest.

APPENDIX D

Time Started ________________
Time Completed________________

Name________________________
ID# ________________________

Pre-Test English 121

BEGIN BY CAREFULLY READING "A WORN PATH" BY EUDORA WELTY. WHEN YOU HAVE READ THE STORY, GO THROUGH THE TEST BELOW AND CARRY OUT THE INSTRUCTIONS AS FULLY AS POSSIBLE (ANY TIME YOU NEED EXTRA SPACE, USE THE BACK OF THE SHEET).

I. Plot. In your own words, describe what happens in the story.(5 points)

Explain what motivates the action of the story. (5 points)

Does the plot of this story rely more on conflict or on character? Explain your answer. (10 points)

II. Meaning. Explain what seems to you to be Eudora Welty's purpose in telling this story. (5 points)

Briefly suggest all the possible meanings which this story conveys to you. (20 points)

Does the story seem simple, ambiguous, complex, pointless to you? Explain. (5 points)

III. Character. Briefly describe the physical appearance of Phoenix Jackson. What significance do you attach to her looks? (10 points)

Make a list of personality traits which Phoenix Jackson demonstrates in the story. Number each trait, and then in the left margin of the text use the numbers to mark the episodes, speeches or scenes which illustrate each trait. (30 points)

IV. Structure. What elements hold this story together for you? (10 points)

List all the episodes or encounters in Phoenix's story. What does each contribute to your understanding? (20 points)

Time Started________________
Time Completed______________

Name______________________
ID#_______________________

Achievement Test English 121

ANALYSIS OF "DRY SEPTEMBER" *Read the story carefully and then follow instructions.*

MEANING. In the text of the story, underscore in black and key (1, 2, etc.) all those passages which are crucial to the meaning of this work. Below explain briefly the meaning each of those keyed passages convey. *25 points possible.*

In one or two carefully developed paragraphs explain the main ideas "Dry September" expresses about human nature and especially interracial relations. *15 points possible.*

Relate the ideas of "Dry September" to other works you have read in this course. *10 points possible.*

CHARACTERIZATION. Listed below are the three most important characters. Using a red marker for 1, a green for 2, and a blue for 3, on the test underscore and key (a) descriptions of these characters, (b) comments about them by other characters or the narrator, and (c) their revealing speeches and actions.

Character 1 Miss Minnie Cooper

Character 2 Henry Hawkshaw

Character 3 John McLendon

15 points possible.

Give a brief resume of Miss Minnie's actions. *5 points possible.*

Referring to the underscored passages, describe the motivation and thought of Henry Hawkshaw. *10 points possible.*

Explain why John McLendon is important in "Dry September" and explain what the author seems to be saying through him. *20 points possible.*

APPENDIX E

RAP SESSIONS

Present:

Major topics discussed:

Leader's Preparation and effectiveness:

Major contributions by:

Problems or difficulties:

______________________ (Date)

______________________ (Signature)

APPENDIX F

EVALUATION OF DISCUSSION LEADER

Name of leader____
(If the discussion leader wishes, he can have the total group's evaluation. Individual evaluations are strictly confidential.)

	Superior	Good	Average	Fair	Poor
Preparation					
Skill at drawing out responses					
Manipulation of conflict					
Reaction to disagreement					
Keeping on the issues					
Review of discussion					

Other comments:

Signed____________________________________

DISCUSSION LEADER'S SELF-EVALUATION

When you have led a rap session, you should evaluate the success of your efforts in a one page statement. Your evaluation should consider the effectiveness of your preparation, your success in stimulating the group to grapple with new or difficult ideas, your handling of controversy, hostility, prejudice or indifference as these may have arisen in the discussion, the major concepts or awarenesses which the group shared, and the course objectives which were served by your discussion. You may, if you wish, see a resume of the evaluation sheets filled out by members of the group.

APPENDIX G

EVALUATION

Name of student being evaluated____
(Your ratings will be strictly confidential, and will not affect the student's grade for English 121.)

GENERAL PARTICIPATION

	Superior	Good	Average	Fair	Poor
General contribution to the group					
Ability to explain ideas					
Reaction to disagreement					
Methods of disagreement with others					
Oral interpretation					
Preparation for sessions					
Use of logical thinking					
Ability to read & analyze selections					
Wise use of the group's time					
Involving others in conversation					

Signed____________________________________

APPENDIX H

PERFORMANCE PROGRAM

After your discussion group concludes all its assignments, it will present a one hour program for two other discussion groups and any visitors who may wish to attend. Each program will be widely publicized on campus.

The performance is included as part of the course to allow for wider dissemination of the ideas from the readings and to provide each student with experiences in oral interpretation. Further, because each student will evaluate and critize one program besides the one he is in, he will of necessity formulate standards for judging the interpretations and public performances of literature.

The two requirements are (1) to make the program unified and (2) to be sure it is good. At the first meeting each group will select a chairman to direct the preparation for the performance. The chairman should be firm in asking others to help him make selections and plan the work. He should be sure that each person has a role which he can handle. He should know that each member of the group understands fully what the performance intends to communicate, knows what all the others are contributing to the total program, and has practiced carefully his part. The performance chairman will be asked to submit a written report of the contribution each group member made in the preparation.

A group may use almost any format, but one which allows full participation by all the group members is preferable. Groups can make a reader's theatre presentation, present a study of unusual characters from the selections, study a complex or interesting theme which appears in several works, review one or two works--with cuttings from the works, plan a panel discussion of some relevant topic, evaluate the philosophical problems of some works, or do whatever the group can dream up.

Evaluation of the program will consider originality, unity of effect, timing, degree of polish, clarity of ideas, and effectiveness of oral interpretation.

The following suggestions will help you with your performance program.

If you plan a series of readings or a cutting from a longer work....
....be sure the ideas are unified and that the unity will be clear to your audience.
....make sure that rehearsals consider how the passages are read so that the interpretation is clear.

...prepare carefully the links which keep the reader from being confused.
...know what others are reading and how it relates to your reading.
...share ideas about effectiveness of reading.
...revise and modify the program as you see the need for change.

If you plan a discussion...
...brainstorm for ideas and select those which reflect the range and significance of the topic.
...be sure you include works which have the most to say on the topic.
...support ideas and views by reading selections from the works.
...be sure that everyone reads some passages as part of the program.
...be sure the readings are introduced properly; i.e., tell what you are reading and how the specific section is introduced.
...work to make the discussion seem spontaneous, even though you will have rehearsed it several times.
...try to avoid loose, rambling talks.
...structure the discussion so that the audience will be able to follow the discussion easily.

If you plan a reader's theatre...
...plan to set the stage carefully for the audience.
...explain the purpose of your presentation.
...rehearse together several times.
...let different ones try the roles till you can be sure your best readers have the longer parts.
...have brainstorming sessions on how to interpret each character and each scene.
...use a stage if possible.

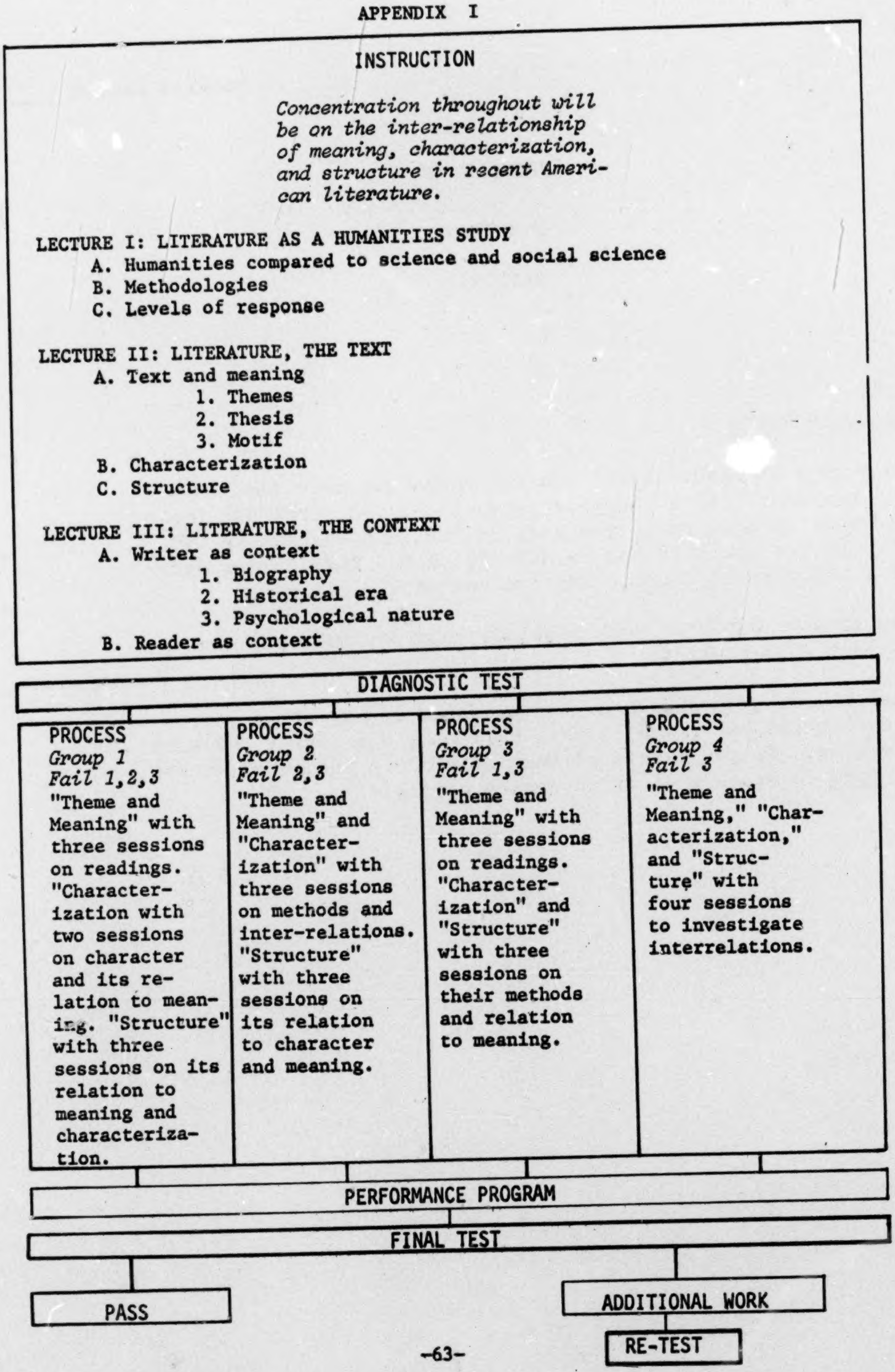
PERFORMANCE EVALUATION

	Superior	Good	Average	Fair	Poor
Total effect of the presentation.					
Presentation Communication of ideas.					
Program's Unity and Coherence.					
Variety and timing.					
Oral interpretation.					
Preparation.					

Other comments:

Signed________________________

APPENDIX I



INSTRUCTION

Concentration throughout will be on the inter-relationship of meaning, characterization, and structure in recent American literature.

LECTURE I: LITERATURE AS A HUMANITIES STUDY
- A. Humanities compared to science and social science
- B. Methodologies
- C. Levels of response

LECTURE II: LITERATURE, THE TEXT
- A. Text and meaning
  - 1. Themes
  - 2. Thesis
  - 3. Motif
- B. Characterization
- C. Structure

LECTURE III: LITERATURE, THE CONTEXT
- A. Writer as context
  - 1. Biography
  - 2. Historical era
  - 3. Psychological nature
- B. Reader as context

DIAGNOSTIC TEST

PROCESS
Group 1
Fail 1,2,3
"Theme and Meaning" with three sessions on readings. "Characterization with two sessions on character and its relation to meaning. "Structure" with three sessions on its relation to meaning and characterization.

PROCESS
Group 2
Fail 2,3
"Theme and Meaning" and "Characterization" with three sessions on methods and inter-relations. "Structure" with three sessions on its relation to character and meaning.

PROCESS
Group 3
Fail 1,3
"Theme and Meaning" with three sessions on readings. "Characterization" and "Structure" with three sessions on their methods and relation to meaning.

PROCESS
Group 4
Fail 3
"Theme and Meaning," "Characterization," and "Structure" with four sessions to investigate interrelations.

PERFORMANCE PROGRAM

FINAL TEST

PASS

ADDITIONAL WORK

RE-TEST

Booklet Number ________

APPENDIX J

Pre-Test

Math 111

GENERAL DIRECTIONS

This is a 45-minute test. Do not spend too much time on any one question. If a question seems to be too difficult, make the most careful guess you can, rather than waste time over it. Do not worry if you do not finish the test. Your score is the number of correct answers you mark.

Use scratch paper to work problems. Do not make any marks in your test booklet.

Mark all answers on the separate answer sheet. Make your answer marks heavy and black. Mark only one answer for each question. If you make a mistake or wish to change an answer, be sure to erase your first choice completely.

Pre-Test

Math 111

SECTION I - Structure

1. Which one of the following is equivalent to $-\frac{3}{4} + \frac{7}{9}$

a) $\frac{1}{36}$ b) $-\frac{55}{36}$ c) $-\frac{37}{49}$ d) $\frac{37}{49}$ e) None of these.

2. In the set of integers we know that $2 + 19 = 19 + 2$

What is the name of the property which this example illustrates.

a) Associative law of addition
b) Associative law of multiplication
c) Closure property of addition
d) Commutative law of addition
e) Inverse property of addition.

3. Which of the following number sets would 3 not be included in.

a) Integers
b) Counting numbers
c) Real Number System
d) Rational numbers
e) Irrational numbers.

4. Find the decimal equivalent of the rational number $\frac{35}{111}$

a) .32
b) .315315315...
c) .315
d) .313131...
e) .31

5. Which of the following examples illustrates the distributive property:

a) $17 \times 99 = 17(100-1) = 1700-17 = 1683.$
b) $17 \times (99 \times 13) = (17 \times 99) \times 13$
c) $17 \times 99 = 1683$
d) $17 + (-17) = 0$
e) $17 + 0 = 17$

6. Which of the following sets of numbers are the integers?

a) $\{0,1,2,3,4, \ldots\}$
b) $\{1,2,3,4, \ldots\}$
c) $\{\ldots, -3, -2, -1, 0,1,2,3,4, \ldots\}$
d) $\{0,1,2,3, \ldots, 9\}$
e) Numbers of the form a/b where $b \neq 0$.

SECTION II - Algebra

Multiple choice. (Only one answer is correct.)
All letters represent real numbers.

1. Remove grouping symbols and add like terms.

$3a - (4b - 7a) - [a + (2a - 3b)]$ The correct answer is:

a) $-3a - 7b$
b) $6ab$
c) $-7a - b$
d) $7a - b$
e) $9a - 7b$

2. Which one of the following is true?

a) $(a - b) + c = (a + c) - (b + c)$
b) $(a + b) \cdot c = a \cdot c + b \cdot c$.
c) $(a \cdot b) - c = (a - c) \cdot (b - c)$
d) $(a \div b) \cdot c = (a \cdot c) \div (b \cdot c)$
e) $(a + b) - c = (a - c) + (b - c)$

3. What is the solution of $3x - 2(1 - x) = 10 + 5x$?

a) $x = -8$
b) $x = 5/6$
c) $x = -1$
d) $x = 12$
e) This equation has no answer

4. What is the solution of $\frac{3x - 1}{6} + \frac{1}{2} = \frac{7x + 4}{3}$?

a) $x = \frac{-6}{11}$

b) $x = 6$

c) $x = \frac{-8}{11}$

d) $x = -\frac{1}{2}$

e) $x = \frac{4}{11}$

5. Simplify by performing indicated operations.

$\left(\frac{-3a^5b}{a^4b^3}\right)^4 =$

(a) $\frac{-3a^4}{b^8}$

(b) $\frac{-81a}{b^2}$

(c) $\frac{81a^5}{b^6}$

(d) $\frac{81a^4}{b^6}$

(e) $\frac{81a^4}{b^8}$

6. Multiply and collect like terms.

$3x(4 - x)\ (2 + x)$

a) $72x^2 + 18x^3 - 9x^4$

b) $24x + 18x^2 - 3x^2$

c) $24x - 18x^2 - 3x^3$

d) $24x + 6x^2 - 3x^3$

e) $24x + 6x^2 + 3x^3$

7. Solve the following equation for h.

$T = 2\pi r^2 + 2\pi rh$

(a) $h = T - r$

(b) $h = \frac{T - 2\pi r^2}{2\pi r}$

(c) $h = T - 2\pi r^2 - 2\pi r$

(d) $h = \frac{T}{2\pi r^2}$

(e) $h = 2\pi r^2 + 2\pi rT$

8. Write an algebraic formula that corresponds to the following word statement.

The force F with which two point masses attract each other is a constant k times the product of the masses m_1 and m_2 divided by the square of the distance r between the masses.

(a) $Fkm_1m_2r^2$

(b) $F = km_1m_2r^2$

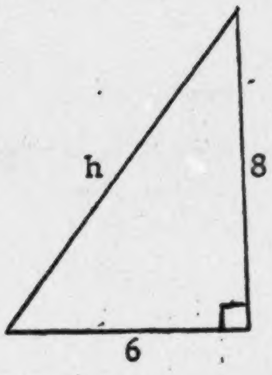
(c) $F = \frac{km_1m_2}{r^2}$

(d) $F = \frac{m_1m_2}{r^2}$

(e) $F = \frac{km_1m_2}{r}$

SECTION III - Geometry

1. Given the following right triangle find the hypotenuse h.

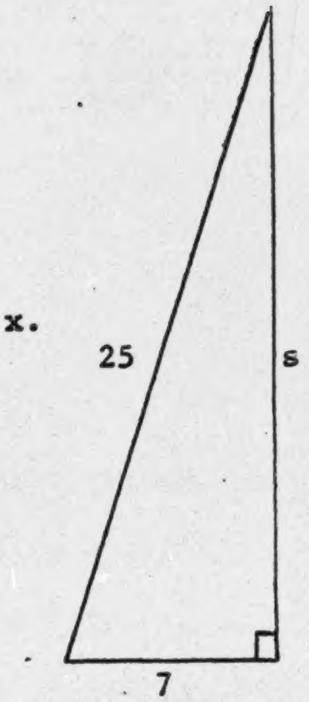


(a) 100 (d) 14
(b) 10 (e) 5.3
(c) 28

2. If $x = \sqrt{800}$, which one of the following is closest in value to x.

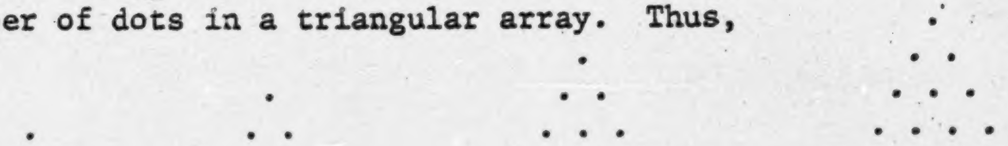
(a) 28.1 (c) 28.3
(b) 28.2 (d) 28.4 (e) 28.5

3. Given the following right triangle, which one of the following is closest in value to the missing side s.



(a) 27.2 (c) 22 (e) 32
(b) 21.6 (d) 24

4. The following numbers are called triangular numbers. They are just the number of dots in a triangular array. Thus,



$T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, etc.

The next two triangular numbers are

(a) $T_5 = 14$, $T_6 = 19$.
(b) $T_5 = 14$, $T_6 = 20$
(c) $T_5 = 15$, $T_6 = 20$
(d) $T_5 = 15$, $T_6 = 21$
(e) $T_5 = 16$, $T_6 = 22$

5. Can the following three numbers be sides of a right triangle and if so, which side is the hypotenuse h.

13, 5, 12.

(a) No, they cannot be sides of a right triangle.
(b) Yes, h = 5
(c) Yes, h = 13
(d) Yes, h = 12.
(e) Yes, any side can be the hypotenuse.

6. Given line segments of lengths 3, 3, and 7, can a right triangle of hypotenuse 7 be formed?

(a) Yes (b) No

SECTION IV - Statistics

Data: 12, 14, 8, 9, 13, 19, 17, 10, 14, 15.

1. Which of the following is the arithmetic mean of the above set of data?

(a) 11.5 (c) 13.5 (e) 14
(b) 12.2 (d) 13.1

2. Which of the following numbers is the median of the above set of data?

(a) 13 (c) 13.5 (e) 12.5
(b) 14 (d) 13.1

3. Which of the following numbers is the mode of the above set of data?

(a) 13 (c) 12 (e) 14
(b) 13.5 (d) 15

4. The range of a set of data is the difference between the highest and lowest value in a set of data. The range of the above set of data is:

(a) 9 (c) 11 (e) 13
(b) 10 (d) 12

5. The standard deviation of a set of data is a measure which tells something about the __________________________of the data. Choose the correct word to fill in the blank.

(a) mean (c) variability (e) compatibility
(b) average (d) flexibility

6. Two tests were given. On the first test the mean was 62 and the standard deviation was 8. On the second test the mean was 60 and the standard deviation was 6. The following are the scores of three students.

	Test 1	Test 2
Tom	70	68
Dick	58	58
Harry	78	72

Using the above means and standard deviations, which student(s) did better on the second test than on the first.

(a) Tom and Dick (c) Dick and Harry (e) Tom
(b) Tom and Harry (d) Dick

7. Listed below are the annual salaries in a small business: 25,000, 12,000, 8,000, 8,000, 7,500, 7,500, 7,500, 7,500, 6,000, 3,500. Find the mean, median, and mode of these salaries. Which one(s) best represents the average salary of workers at this business. Mark the most correct answer.

(a) Mean
(b) Median
(c) Mode
(d) Median or Mode since they are both the same.
(e) Mean or Median

SECTION V - Probability

1. Suppose a single die, perfectly balanced, is rolled. There are six equally likely outcomes: 1, 2, 3, 4, 5, 6. What is the probability that an even number will be rolled:

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$ (e) $\frac{3}{21}$

2. Suppose that a couple plan to have three children. What is the probability that at least two of the children will be boys?

(a) $\frac{3}{8}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$ (e) 0

3. Suppose that two fair dice are rolled. What is the probability that the sum of the number of spots showing on the two dice is 7?

(a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{3}{36}$ (d) 1 (e) $\frac{1}{2}$

4. Suppose a fair coin is tossed and a single fair die is rolled. What is the probability that the coin will turn up tails and the die will turn up a three?

(a) $\frac{1}{8}$ (b) $\frac{2}{3}$ (c) $\frac{1}{12}$ (d) $\frac{1}{4}$ (e) $\frac{1}{36}$

5. Suppose an opaque jar contains 6 red marbles and 4 green marbles. Suppose the marbles are randomly mixed and you reach in and draw a marble. Then without replacing the first marble you draw a second marble. What is the probability of drawing 2 red marbles?

(a) $\frac{3}{10}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{9}$ (e) $\frac{2}{3}$

6. Suppose that a single fair die is rolled. What is the probability that a 1 or 5 will be rolled?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{36}$ (d) $\frac{1}{2}$ (e) $\frac{1}{8}$

Booklet Number________________

Name________________________

Answer Sheet for
Pre-test Math 111

SECTION I - Structure

___1. ___2. ___3. ___4. ___5. ___6.

SECTION II - Algebra

___1. ___2. ___3. ___4. ___5. ___6. ___7. ___8.

SECTION III - Geometry

___1. ___2. ___3. ___4. ___5. ___6.

SECTION IV - Statistics

___1. ___2. ___3. ___4. ___5. ___6. ___7.

SECTION V - Probability

___1. ___2. ___3. ___4. ___5. ___6.

APPENDIX K

MATHEMATICS 111

Individualized study for ____

A check mark indicates that the pre-test shows you are deficient in the following areas.

Section I -- Structure ☐
" II -- Algebra ☐
" III -- Geometry ☐
" IV -- Statistics ☐
" V -- Probability ☐

If you have no check marks in the boxes above then you have completed this course.

If you have one to four check marks then read the To the Student section of the book, Mathematics, the Essence of Learning to determine the grade you wish to receive in this course. Then follow the outline for this trimester to make that grade by studying those areas above in which you were found to be deficient.

If you have five check marks then the pre-test indicates you should take

Option 1. ☐
" 2. ☐

Information concerning options one and two may be found in the To the Student section of Mathematics, the Essence of Learning for those deficient in "all 5 of the areas."

APPENDIX L

Outline for
Mathematics 111
Spring Trimester, 1971

TEXTBOOK, Mathematics, the Essence of Learning

May 31 - Introduction
June 2 - Pre-test
June 4 - Results of Pre-test

SECTION I - Structure

June 7 - Lesson 1, pp. 1-9
9 - Lesson 2, pp. 10-20
11 - Discussion and problem solving session
14 - Test over Section I

SECTION II - Algebra

June 16 - Lesson 3, pp. 22-28
18 - Lesson 4, pp. 29-37
21 - Lesson 5, pp. 38-43
23 - Lesson 6, pp. 44-49
25 - Discussion and problem solving session
28 - Test over Section II

SECTION III - Geometry

June 30 - Lesson 7, pp. 51-60
July 2 - Holiday
5 - Lesson 8, pp. 61-68
7 - Discussion and problem solving session
9 - Test over Section III

SECTION IV - Statistics

July 12 - Lesson 9, pp. 70-78
14 - Lesson 10, pp. 79-83
16 - Lesson 11, pp. 84-92
19 - Discussion and problem solving session
21 - Test over Section IV

SECTION V - Probability

July 23 - Lesson 12, pp. 94-103
26 - Lesson 13, pp. 104-111
28 - Lesson 14, pp. 112-121
30 - Discussion and problem solving session
Aug. 2 - Test over Section V

APPENDIX M

MATHEMATICS INSTRUCTIONAL MATERIAL

PREFACE

This book is intended for students at Oklahoma Christian College as part of their foundation study in the area of mathematics.

The material is designed so that persons with little or no training in mathematics can easily understand the basic principles and follow the explanations.

The topics selected for discussion include five basic areas of mathematics. These are structure, algebra, geometry, statistics, and probability. The first two topics are intended to give the student the basic mathematical tools to handle problems that arise in mathematical applications as well as simple applications in other fields. The last three topics are designed to be as interesting as possible while at the same time to give some idea of how mathematics is used in various applications.

Lessons 1 through 6 present the basic structure of the number system and put these rules to practical use in solving equations and setting up simple word problems. Lessons 7 and 8 are mostly a study of the Pythagoreans and the famous right triangle theorem proved by Pythagoras. Lessons 9 through 14 give some of the fundamental applications which are involved in the area of statistics and probability.

TO THE STUDENT

This book and course is designed for all students at Oklahoma Christian College. It is recognized at the outset that not all students are the same. Everyone differs in his ability to do mathematics. Some have had a very high level of training in high school and know how to apply their knowledge to problems in science, social science, and technology. Others have had a very weak background in mathematics and need to proceed at a much slower pace in order to learn the basic principles needed to function in society. Still others are somewhere in between. They need to be refreshed or taught some areas of mathematics but also remember many principles taught in high school.

The purpose of this course is to provide you with an individualized study by giving credit for those areas which you already know and then give you an opportunity to study those areas which you do not know or have forgotten.

To provide you with an individualized study requires that you take a pre-test to determine which of the five areas in this book you already know. The procedure goes as follows. At the beginning of the trimester a pre-test is given over structure of the number system, algebra, geometry, statistics, and probability.

ON THE PRE-TEST IF YOU ARE DEFICIENT IN:

none of these areas	You receive a grade of A in the course and 1 hour of credit in mathematics.

one area, you have one of two options	1. You receive a grade of B in the course and 1 hour of credit in mathematics. 2. You receive a grade of A in the course by passing a test over the area in which you are deficient at the 80% level by the same procedure as those students deficient in more than one area.
2, 3, or 4 areas, you take one of the following options.	1. You receive a grade of A in the course and 1 hour credit in mathematics by completing the sections in which you are deficient and passing tests over all these areas at the 80% level. 2. You receive a grade of B in the course and 1 hour credit in mathematics by completing all but one of those sections in which you are deficient and passing tests over these sections at the 80% level. 3. You receive a grade of C in the course and 1 hour credit in mathematics by completing all but one of those sections in which you are deficient and passing tests over these sections at the 70% level. 4. You receive a grade of F in the course if you fail to achieve options 1, 2, or 3.
all 5 of the areas then you have one of two options.	1. Although you were deficient in all five sections you were near enough to passing in most areas to attend regular classes of each section as given by the class outline until you have passed a minimum of 4 sections at the 70% level. 2. Purchase the following materials at the bookstore: Fundamentals of Arithmetic by Eraut. Then meet at the regularly scheduled time in the room announced with the student tutor. After this workbook is completed and a proficiency test is passed you then receive credit for the first two sections of this book, that is, the structure and algebra sections. You then must complete the sections on geometry, statistics, and probability. If you finish the workbook before all these sections are completed in regular classroom sessions, you may complete as many of these sections as possible during the trimester. In the event you do not complete the workbook or all of the sections of this book and you are making satisfactory progress towards completion you receive a grade of W in the course and must reenroll another trimester until a minimum of 4 sections is completed at the 70% level. If you are not making satisfactory progress by the end of the trimester you receive a grade of F in the course.

There are two ways to complete a section in which you are deficient.

The first way is to attend classes over the deficient section and pass a test over this section as given by the class outline. A class outline which specifies the dates of the classes and tests will be given you at the beginning of each trimester.

The second way to complete a section is to work through each lesson of the section on your own and complete all problem sets at the end of the lessons. Then by demonstrating to the instructor that you have completed this section you will be given a test over this section. If you pass the test then the section is completed. If you fail the test you then must attend classes and take the regularly scheduled test given by the class outline over this section. In order to complete a section in this way you must do so before regular classes are offered on this section.

In the event you fail a regularly scheduled test given by the class outline or do not pass it at the level you want, you may retake this test during the final test period at the time scheduled for this class. However, you may take only one test at that time. This means you cannot complete two or more sections during the final test period.

CONTENTS

SECTION I, Structure of the Number System

Lesson 1 1
Lesson 2 11

SECTION II, Algebra 23

Lesson 3 25
Lesson 4 33
Lesson 5 43
Lesson 6 49

SECTION III, Geometry 55

Lesson 7 57
Lesson 8 67

SECTION IV, Statistics 75

Lesson 9 77
Lesson 10 87
Lesson 11 93

SECTION V, Probability 103

Lesson 12 105
Lesson 13 115
Lesson 14 123

SECTION I

Structure of the Number System

LESSON 1

Structure of the Number System

In this section we want to learn about some of the basic properties which characterize our number system. Many of the properties are familiar to us. However, most of us know very little about the names of these properties or the overall structure of our number system. Thus, we shall explore some of the basic principles by first studying about these properties in a completely abstract mathematical system. The use of the abstract system should explain to us the basic properties of our number system without confusing us about those with which we are already familiar.

Let us define an abstract system composed of the set of elements:

$$\{\alpha, \beta, \gamma, \delta\}.$$

Let us call this set S; that is,

$$S = \{\alpha, \beta, \gamma, \delta\}.$$

We next define an operation on the set S which we shall call staration and denote this by the symbol *. Staration is defined by the following table:

*	α	β	γ	δ
α	α	β	γ	δ
β	β	γ	δ	α
γ	γ	δ	α	β
δ	δ	α	β	γ

The operation * is a binary operation since for any two elements of the set S it associates with it a single element of S.

To "star" two elements of S we find the first element in the vertical column at the left and the second element in the horizontal row across the top and then find where the column and row intersect within the table. For examples we have:

$$\beta * \gamma = \delta$$

$$\delta * \delta = \gamma$$

$$\alpha * \gamma = \gamma.$$

We now investigate the basic properties of the system defined by staration on the set S. Some of the most important properties are closure, commutative, associative, identity, and inverses.

DEFINITION 1.1: Closure. In general, a set B is said to be closed under a binary operation if that operation associates a unique element of the set B with any two elements from B.

Thus, for the set S defined above and the operation staration we see that whenever any two elements from S are starred the result is a unique element of the original collection S of elements. In other words, there is one and only one answer whenever two elements of S are starred, and this answer is always one of the elements of S. Therefore, from the definition of closure we say that the set S is closed under the operation of staration.

As another example, let us consider C to be the set of counting numbers which we use in counting. Then

$$C = \{1, 2, 3, 4, 5, \cdot \cdot \cdot\}$$

where the three dots are used to indicate that C is an infinite set and not all of its elements can be listed. Of course we know that there is no largest counting number and C is sometimes referred to as the set of positive integers.

Let us now examine C under the ordinary operation of addition for closure. That is, we are asking if C is closed under the operation of addition. We do not need an addition table since all of us have it memorized. We see that the addition of any two counting numbers is again another counting number and C is closed under addition.

Is the set C closed under subtraction? That is, is the difference of any two counting numbers always a counting number?

Notice the importance of the word "any" in the definition of closure.

DEFINITION 1.2: Commutative. A set B is said to have the commutative property with respect to a binary operation # (# stands for any binary operation) if for all elements of B

$$a \# b = b \# a$$

where a and b are any members of the set B.

Another way of saying that a set has the commutative property with respect to a binary operation is that the result obtained by combining any two elements of the set under that operation does not depend upon the order in which these elements are combined.

Consider the set S and the operation staration. Is S commutative with respect to the operation * ? That is, given any two elements of S, call them a and b, does $a * b = b * a$? From the *- table we can check and see that the answer is yes for any two elements we choose. Therefore, we say S is commutative with respect to staration.

The set of counting numbers is commutative under the operations of addition and multiplication as the following examples illustrate:

$3 + 5 = 5 + 3 = 8$ $\quad$ $5 \cdot 13 = 13 \cdot 5 = 65$

$13 + 71 = 71 + 13 = 84$ $\quad$ $3 \cdot 7 = 7 \cdot 3 = 21$

From these examples one might infer that all sets are commutative under various binary operations. However, consider the following:

PROBLEM 1.3: Show that subtraction of counting numbers is not commutative.

Solution: A single counterexample is sufficient to show that a property does not hold since in order to satisfy the property it must do so for all numbers.

$$9 - 6 = 3 \quad \text{but} \quad 6 - 9 = -3$$

Therefore: $9 - 6 \neq 6 - 9$ and the counting numbers are not commutative under subtraction.

DEFINITION 1.4: Associative. A set B is said to have the associative property with respect to the operation # if for all elements of B

$$a \# (b \# c) = (a \# b) \# c$$

where a, b, and c are any members of B.

The associative property deals with the order in which we apply the binary operation. On the left hand side of the above we combined b and c first and then combined a with the result of b and c. On the right hand side we combined a and b first and then combined the result with c.

Checking the set S and the operation staration we find that $a \star (b \star c) = (a \star b) \star c$ where a, b, and c may be replaced by any elements of S. One such case is the following example:

$$\gamma \star (\beta \star \delta) = \gamma \star \alpha = \gamma$$

while $$(\gamma \star \beta) \star \delta = \delta \star \delta = \gamma.$$

Therefore:

$$\gamma \star (\beta \star \delta) = (\gamma \star \beta) \star \delta.$$

PROBLEM 1.5: Show that the set C of counting numbers is not associative with respect to division.

Solution: Again one counterexample is sufficient.

$$16 \div (12 \div 3) \neq (16 \div 12) \div 3,$$

because $16 \div (12 \div 3) = 4$ and $(16 \div 12) \div 3$ is not a counting number and thus cannot be equal to 4.

DEFINITION 1.6: Identity. An identity for a set B with respect to the binary operation # is an element of B (say e) such that

$$a \# e = e \# a = a$$

for all elements a of B.

Consider the counting numbers under multiplication. Does there exist a counting number e such that

$$a \cdot e = e \cdot a = a$$

for all counting numbers a? Yes, the counting number 1 has this property and 1 is an element of C. Therefore, the set C has an identity, namely 1, under the operation of multiplication.

Does the set S under the operation of staration have an identity element? Can you find the identity?

We now define the set of integers I.

$$I = \{\cdot\ \cdot\ \cdot, -4, -3, -2, -1, 0, 1, 2, 3, \cdot\ \cdot\ \cdot\}.$$

The set of integers is sometimes divided into three subsets, the positive integers or counting numbers, zero, and the negative integers. The set of integers has an identity for both addition and multiplication. These are 0 and 1 respectively.

The last basic property of our number system considered here is inverses.

DEFINITION 1.7: Inverses. A set B is said to have inverses with respect to the binary operation # if for each a an element of B there exists an element $\overline{a}$ of B such that

$$a \# \overline{a} = \overline{a} \# a = e$$

where e is the identity for #.

The set C of counting numbers does not have inverses with respect to the operation multiplication. For what counting number multiplied times 2 will give 1 the identity for C?

$$2 \cdot \overline{a} = \overline{a} \cdot 2 = 1?$$

We see that $a = \frac{1}{2}$ will work but $\frac{1}{2}$ is not a counting number and not an element of C. Therefore, C does not have inverses with respect to multiplication.

Does I the set of integers have inverses with respect to addition? Yes, we see that given any integer a there exists another integer called -a such that $a + (-a) = 0$ where 0 is the identity for addition of integers. Examples are:

$$3 + (-3) = 0$$

$$37 + (-37) = 0.$$

If a set of elements B satisfies all five of the above definitions under # then we say that set B forms a commutative group under #. Thus, a set B under the binary operation of # is a commutative group with respect to # if and only if the following five properties are satisfied:

1. Closure with respect to #.
2. Commutative with respect to #.
3. Associative with respect to #.
4. B has an identity element with respect to #.
5. Each element of B has an inverse with respect to #.

Exercises for Lesson 1.

1. Find the answer to each of the following from the table given in this section.

(a) $\beta * \beta =$

(b) $\delta * \alpha =$

(c) $\gamma * \beta =$

2. Which of the following sets are closed?

(a) C with respect to $\cdot$.

(b) I with respect to $-$.

(c) I with respect to $\div$.

(d) C with respect to $\div$.

3. Does the set C have an identity with respect to the operation +?

4. From the table given in this section find the inverses of

(a) α (c) γ

(b) β (d) δ

5. Show that the set I of integers forms a commutative group with respect to +.

6. State the definition or definitions which justify each of the following statements:

(a) $5 + 7 = 7 + 5$

(b) $(5 + 1) + 4 = 5 + (4 + 1)$

(c) $(5 + 9) + 3 = 3 + (5 + 9)$

(d) $(8 \cdot 9) \cdot 3 = 8 \cdot (9 \cdot 3)$

(e) $\alpha * \beta = \beta * \alpha = \beta$

(f) $\beta * \delta = \delta * \beta = \alpha$

7. Consider the binary operation ¢ on the set $G = \{1, 2, 3, 4, 6, 12\}$, defined by the following table:

¢	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

(a) Is ¢ closed over G?

(b) Is ¢ commutative over G?

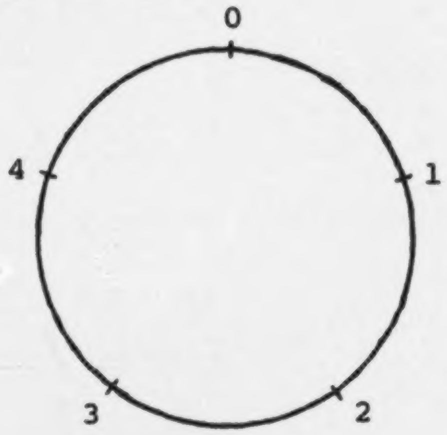
(c) Is ¢ associative over G?

(d) Is there an identity for ¢?

(e) Does each element of G have an inverse with respect to ¢? Find the inverses, if any?

8. Prepare an addition table for a 5-hour clock.

+	0	1	2	3	4
0					
1					
2					
3					
4					



9. Determine if the numbers on a 5-hour clock, under the addition of problem 8, form a commutative group?

10. Using the properties of a group prove the following:

(a) The identity element of a group G is unique.

(b) The inverse element of any element of a group G is unique.

(c) If a and b are elements of a group G, and

a # b = a # c then b = c.

Answers for Lesson 1.

1. (a) γ (b) δ (c) δ
2. (a) and (b) are closed.
3. No
4. (a) α (b) δ (c) γ (d) β
5. Hint: Give examples which illustrate each of the five properties of a group.
6. (a) Commutative with respect to addition
   (b) Associative and commutative with respect to addition
   (c) Commutative with respect to addition
   (d) Associative with respect to multiplication
   (e) Commutative and identity with respect to staration
   (f) Commutative and inverse with respect to staration
7. (a) Yes (b) Yes
   (c) Yes, example: (4 ¢ 2) ¢ 3 = 4 ¢ (2 ¢ 3) = 1
   (d) Yes, 12 (e) No, 12 is an inverse of 12

8.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

9. (a) Closure - yes (b) Commutative - yes
   (c) Associative - yes Example: $(2 + 3) + 4 = 2 + (3 + 4) = 4$
   (d) Identity - yes, 0
   (e) Inverse - yes, Example: The inverse of 1 is 4. $1 + 4 = 0$
   Therefore, the set is a commutative group.
10. (a) To prove that the identity is unique, suppose G has two identities, say e and e'.

    For every a that is an element of G,

$$ae = a$$
$$ae' = a,$$

    Since e is an element of G,

$$ee' = e$$
$$ee = e,$$

which implies

$$ee' = ee$$

and

$$e' = e.$$

LESSON 2

Structure of the Number System

From the first lesson we have learned about many properties of our number system, but in doing so we have limited ourselves to two basic sets of numbers; the counting numbers

$$C = \{1, 2, 3, 4, 5, \cdots\},$$

and the integers

$$I = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}.$$

Just as the counting numbers are insufficient for our needs, (we need more numbers than just counting numbers in order to function in the present world), so also are the integers. For example, what integer tells us the shooting ability of a given basketball player? We quickly see that no such integer exists. However, with two integers we can describe the percentage of baskets the player is making. We might say he is averaging 7 baskets for each of 12 tries at the basket. What is usually done though is to describe the player's shooting ability by dividing 12 into 7 which gives .5833 · · ·, and then say he is shooting 58%. Thus, we are able to interpret that on the average of 100 shots at the basket the player will make a basket 58 times. Even though an integer may not describe a basketball player's shooting ability a number of the type 58/100 or 58% makes it perfectly clear. These types of numbers are called rational numbers and they include the integers just as the integers include the counting numbers. We now give the definition of a rational number.

DEFINITION 2.1: A rational number is a number of the form a/b where a and b are integers with b not equal to zero.

Many people think of a rational number as a fraction whose value is a positive number less than one. However, it may be greater than one, it may be negative, or it may be an integer. Many times it may be of some advantage to think of it as just being the ratio of any two integers except that the bottom one cannot be zero. Some examples of rational numbers are:

$$-10,\ 25/7,\ -4/5,\ \sqrt{36},\ 1,\ 45\%.$$

How does -10, $\sqrt{36}$, and 1 fulfill the definition of being the ratio of two integers?

Before we can work with the rational numbers we need to know how to define the binary operations of addition, subtraction, multiplication and division. In defining these operations we run into another problem of representation of rationals. That is, we all agree that 3/4 and 9/12 represent the same rational number but we can't be so sure that 93/124 is the same as 3/4. Thus, we need to be able to determine what is meant by equality of rational numbers.

DEFINITION 2.2: Equality of rationals. $\frac{a}{b} = \frac{c}{d}$ if and only if ad = bc. From this definition of equality it is easy to determine that

$$3/4 = 93/124$$

because $3 \cdot 124 = 372$ and $4 \cdot 93 = 372$.

It is also equally easy to see that

$$3/4 \neq 158/212$$

because $3 \cdot 212 = 636$ and $4 \cdot 158 = 632$.

With this definition of equality of two rational numbers we are now ready to define binary operations on the rationals.

DEFINITION 2.3: Addition of rationals.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Example: $$\frac{9}{10} + \frac{3}{4} = \frac{4 \cdot 9 + 3 \cdot 10}{4 \cdot 10} = \frac{66}{40} = \frac{33}{20}$$

DEFINITION 2.4: Multiplication of rationals.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Example: $$\frac{9}{10} \cdot \frac{3}{4} = \frac{27}{40}$$

The inverses of these two operations are subtraction and division defined in the next two definitions.

DEFINITION 2.5: Subtraction of rationals.

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - c \cdot b}{bd}$$

Example: $$\frac{25}{7} - \frac{13}{5} = \frac{25 \cdot 5 - 13 \cdot 7}{7 \cdot 5} = \frac{34}{35} .$$

DEFINITION 2.6: Division of rationals.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c} = \frac{ad}{bc} .$$

Sometimes the division operation is illustrated as

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad \text{or} \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

and we say to divide rationals one inverts the divisor and then multiplies according to the rule of multiplication of rationals.

$$-\frac{3}{4} \div \frac{13}{6} = \frac{-3/4}{13/6} =- \frac{3}{4} \cdot \frac{6}{13} = \frac{-18}{52} = \frac{-9}{26}$$

With the binary operations now defined we want to know about the basic structure of the rational numbers. That is, do they satisfy the commutative and associative properties for addition and multiplication?

Do the rationals have an identity for addition and multiplication?

PROBLEM 2.7: Prove that the rationals are commutative with respect to addition.

Proof: We need to show that

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b} .$$

By definition 2.3,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and

$$\frac{c}{d} + \frac{a}{b} = \frac{cb + da}{db} .$$

Now, bd = db because b and d are integers and the integers are commutative over multiplication. For the same reason, bc = cb and ad = da. Therefore:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = \frac{da + cb}{db} ,$$

but

$$\frac{da + cb}{db} = \frac{cb + da}{db} = \frac{c}{d} + \frac{a}{b}$$

because the integers are commutative under addition.

Hence

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b} .$$

We now define the first basic property of our number system which involves a combination of two binary operations. This is called the distributive property.

DEFINITION 2.8: The binary operation multiplication is said to be distributive with respect to the binary operation addition if

$$a\cdot(b + c) = a\cdot b + a\cdot c.$$

Example: Evaluate $4\cdot(7+13)$.

The distributive property tells us we can evaluate this expression in either of two ways. We may add 7 and 13 and then multiply as in

$$4\cdot(7+13) = 4\cdot(20) = 80,$$

or we may multiply the 4 and 7, then the 4 and 13 and add the two products as in

$$4\cdot(7 + 13) = 4\cdot7 + 4\cdot13 = 28 + 52 = 80.$$

The result will be the same.

The distributive property of multiplication over addition is quite useful to us in algebra because it allows us to make such statements as:

$$2(x + y) = 2x + 2y$$

$$(a + b)\cdot(a - b) = a^2 - b^2.$$

It is also used in multiplication in the elementary school. Consider the problem $9\cdot56$. The distributive property is used by thinking of $9\cdot56$ as $9\cdot(50 + 6)$ and usually performed in this fashion.

$$\begin{array}{r l}
56 & \\
\underline{9} & \\
54 & = 9\cdot6 \\
\underline{450} & = 9\cdot50 \\
504 & = 9\cdot6 + 9\cdot50
\end{array}$$

Another application of the distributive law is found in the exercises.

We now give an alternate definition of a rational number in terms of decimal notation.

DEFINITION 2.9: A rational number is a number such that when written in decimal notation either terminates or is a repeating decimal.

Thus we see that the following numbers are rational numbers by this definition.

$$\frac{1}{3} = .3333 \cdots$$

$$\frac{6}{5} = 1.2.$$

$$\frac{-31}{37} = -.837837837 \cdots$$

It is also easy to visualize numbers which are non-repeating decimals. These numbers are called irrational numbers. Some examples are

$$\sqrt{2} = 1.414213562 \cdots$$

$$\pi = 3.141592653 \cdots$$

$$\sqrt[4]{75} = 2.942830956 \cdots$$

The rational numbers together with the irrational numbers make up what is sometimes called the Real Number System or the real numbers. Thus, if a number X is a real number, it is either a repeating decimal or a non-repeating decimal. That is, it is either a rational number or an irrational number.

In a non-repeating decimal we see that there is no way of predicting what digit will occur in the next decimal place. For example, in the problem $\frac{-31}{37}$ it is easy to see what digit will occupy the tenth decimal place. It is an 8. But for π, what digit will occupy the tenth place?

Definitions 2.1 and 2.9 give us two distinct ways of representing rational numbers, yet the numbers are the same numbers regardless of the notation used. Thus, we need to be able to convert from numbers of the form a/b to decimals, and from decimal notation to the form a/b where a and b are integers.

The first of these conversions is quite easy as the next problem illustrates:

PROBLEM 2.10: Determine the decimal equivalent of 8/33.

Solution: In order to solve this problem we divide 33 into 8.

$$\begin{array}{r} .242424\cdots \\ 33\,\overline{)\,8.000000\cdots} \end{array}$$

Thus

$$\frac{8}{33} = .242424\cdots$$

is a repeating decimal which repeats the block 24.

The technique of converting a repeating decimal to a fraction of the form a/b where a and b are integers is given by the following examples:

PROBLEM 2.11: Convert the repeating decimals $3.6666\cdots$ and $.019019019\cdots$ to rationals of the form a/b.

Solution: In the first number $3.6666\cdots$ we see that one digit 6 is repeated over and over. So let

$$N = 3.6666\cdots$$

then

$$\begin{array}{rl} 10N &= 36.6666\cdots \\ -N &= -3.6666\cdots \\ \hline 9N &= 33. \end{array}$$

Hence

$$N = 33/9 = 11/3.$$

In the second number $.019019019\cdots$, we see that the block 019 is repeated over and over. So let

$$N = .019019019\cdots$$

then

$$\begin{array}{rl} 1000N &= 19.019019019\cdots \\ -N &= -.019019019\cdots \\ \hline 999N &= 19 \end{array}$$

Hence

$$N = 19/999.$$

Generally, in the conversion process if one digit is repeated over and over we multiply the number by 10. If a two digit block is repeated we multiply the number by 1000 and so forth, in order to convert the number to the quotient of two integers.

Exercises for Lesson 2.

1. Prove the commutative and associative laws for multiplication in the set of rational numbers.

2. Definition 2.3 tells us how to add two rational numbers. Suppose we change the definition of addition to

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d} .$$

Perform the following using this definition.

(a) $\frac{3}{5} + \frac{2}{7} =$ (b) $\frac{3}{1} + \frac{5}{1} =$ (c) $\frac{25}{12} + \frac{6}{12} =$

3. Using the definition of addition in problem 2 above consider the following example:

A baseball player made 2 runs out of 5 times at bat on Friday, and 3 runs out of 7 times at bat on Sunday. His combined record for the two days would be meaningfully given by the following addition:

$$\frac{2}{5} + \frac{3}{7} = \frac{5}{12} .$$

This example gives us a concrete interpretation of the type of addition defined in problem 2.

(a) Explain any fallacies you find in using this addition for everyday use.

(b) Give a concrete interpretation of addition as defined normally in definition 2.3.

4. Show that the set Q of rationals form a group with respect to the operation +.

5. Does the set Q of rationals form a group with respect to multiplication? Why?

6. Perform the following operations:

(a) $\frac{2}{-5} + \frac{-2}{-3} =$

(b) $\frac{-6}{5} \cdot \frac{7}{2} =$

(c) $\frac{2}{3} \div \frac{7}{-5} =$

(d) $\left(\frac{4}{5} + \frac{7}{3}\right) \div \frac{5}{6} =$

(e) $\left(\frac{25}{7} - \frac{31}{10}\right) + \frac{14}{35} =$

(f) $\frac{12}{50} \cdot \frac{-5}{6} \cdot \frac{-5}{2} =$

7. In definition 2.1 of a rational number we went to some lengths in specifying that the denominator of a rational number cannot be zero. Give a meaning for each of the following and then explain why division by zero is prohibited.

(a) $\frac{0}{6}$

(b) $\frac{0}{0}$

(c) $\frac{a}{0}$, $a \neq 0$

(d) $\frac{a}{a}$, $a \neq 0$

(e) $\frac{\frac{1}{3}}{0}$

8. Show that addition is not distributive over multiplication by giving an example.

9. The distributive property is useful in developing shortcuts in multiplication. Thus

$$8 \cdot 99 = 8(100 - 1) = 800 - 8 = 792.$$

Find the following products by using this method:

(a) $7 \cdot 79$

(b) $5 \cdot 58$

(c) $98 \cdot 9$

(d) $107 \cdot 99$

10. Given the problem $5 + 3 \cdot 4 = ?$, three students gave the following answers:

(a) $5 + 3 \cdot 4 = 8 \cdot 4 = 32$

(b) $5 + 3 \cdot 4 = 5 + 12 = 17$

(c) $5 + 3 \cdot 4 = (5 + 3) \cdot (5 + 4) = 8 \cdot 9 = 72.$

Which is correct? State what is wrong with the incorrect ones.

11. Show that the rational numbers are closed with respect to addition by using definition 2.9.

12. Find the decimal equivalents of the following:

(a) $\frac{3}{7}$ (b) $\frac{5}{8}$ (c) $\frac{31}{9}$ (d) $\frac{17}{13}$

13. Write the following in the form a/b where a and b are integers.

(a) .99999 · · · (c) 16.23 (e) .31313131 · · ·

(b) 31% (d) 5.136136136 · · ·

14. Explain the difference between the rational numbers

.5 and .49999 · · ·.

15. Prove that the following statements are true.

(a) $\frac{11}{391} \neq \frac{21}{781}$ (b) $\frac{3}{41} \neq \frac{129}{1762}$

Answers for Lesson 2.

2. (a) $\frac{5}{12}$ (b) $\frac{8}{2}$ (c) $\frac{31}{24}$
4. Closure Example: $\frac{3}{4} + \frac{2}{3} = \frac{17}{12}$ $\frac{17}{12}$ is a rational number.

Commutative Example: $\frac{3}{4} + \frac{2}{3} = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$

Associative Example: $\left(\frac{3}{4} + \frac{2}{3}\right) + \frac{1}{2} = \frac{3}{4} + \left(\frac{2}{3} + \frac{1}{2}\right) = 2$

Identity Example: $\frac{32}{178} + 0 = \frac{32}{178}$

Inverse Example: $\frac{32}{178} + \left(\frac{-32}{178}\right) = 0$

5. No, because 0 does not have an inverse element in Q.
6. (a) $\frac{4}{15}$ (b) $\frac{-21}{5}$ (c) $\frac{-10}{21}$ (d) $\frac{94}{25}$ (e) $\frac{61}{70}$ (f) $\frac{1}{2}$
7. (a) 0 (b) undefined (c) undefined (d) 1 (e) undefined
Zero is prohibited because suppose $\frac{a}{0} = b$. Then $a = b \cdot 0 = 0$. But in c)-part, $a \neq 0$. Therefore, we have a contradiction.
8. $\frac{1}{2} + \left(\frac{1}{4} \cdot \frac{2}{3}\right) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
Now using the distributive postulate ,
$\frac{1}{2} + \left(\frac{1}{4} \cdot \frac{2}{3}\right) = \left(\frac{1}{2} + \frac{1}{4}\right) \cdot \left(\frac{1}{2} + \frac{2}{3}\right) = \frac{3}{4} \cdot \frac{7}{6} = \frac{7}{8}$
which is incorrect?

9. (a) $7\cdot 79 = 7\cdot(80 - 1) = 560 - 7 = 553$
   (b) $5\cdot 58 - 5\cdot(60 - 2) = 300 - 10 = 290$
   (c) $98\cdot 9 = (100 - 2)\cdot 9 = 900 - 18 = 882$
   (d) $107\cdot 99 = (100 + 7)(100 - 1)$
   $= 10{,}000 - 100 + 700 - 7$
   $= 10{,}000 + 600 - 7$
   $= 10{,}593$

10. (a) Incorrect, since multiplication precedes addition.
    (b) Correct (c) Incorrect, see problem 8.

12. (a) .42857142857 · · · (b) .625.
    (c) 3.4444 · · · (d) 1.307692307692 · · ·

13. (a) 1 (b) $\frac{31}{100}$ (c) $\frac{1623}{100}$ (d) $\frac{5131}{999}$ (e) $\frac{31}{99}$

SECTION II

Algebra

LESSON 3

Algebra

To begin our study of algebra we will assume that all the numbers we are working with belong to the real number system R. The real numbers R are comprised of the rationals and irrationals which means that some real numbers are positive, some negative, and one is zero.

The real numbers form a commutative group under the operation of addition. If zero is omitted, the real numbers also form a commutative group under the operation of multiplication. (For the definition of a commutative group see lesson 2 of Structure of the Number System). Also, the real numbers satisfy the distributive property of multiplication over addition.

For convenience we now list the postulates of the real number system R. Let a, b, and c be arbitrary elements of R.

3.1. $a + b$ is a unique element of R (closure for addition)

3.2. $a + b = b + a$. (commutative law of addition)

3.3. $(a + b) + c = a + (b + c)$. (associative law of addition)

3.4. There exists a real number 0 such that $a + 0 = 0 + a = a$ for all a in R. (0 is called the identity element for addition)

3.5. For each real number a of R there exists another real number $-a$ with the result that $a + (-a) = (-a) + a = 0$. ($(-a)$ is called the additive inverse or negative of a)

3.6. $a \cdot b$ is a unique element of R. (closure for multiplication)

3.7. $a \cdot b = b \cdot a$. (commutative law of multiplication)

3.8. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. (associative law of multiplication)

3.9. There exists a real number 1 such that $a \cdot 1 = 1 \cdot a = a$ for all a of R. (1 is called the identity element for multiplication)

3.10. For each real number a of R, $a \neq 0$ there exists another real number $\frac{1}{a}$ with the result that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1.$$

($\frac{1}{a}$ is called the multiplicative inverse of a).

3.11. $a \cdot (b + c) = a \cdot b + a \cdot c$. (distributive law of multiplication over addition)

The eleven postulates given above are known as the field postulates of the real number system. These are basic in learning how to function with the real numbers. In addition to the field postulates, the real numbers satisfy many other useful properties which make them the fundamental set of numbers in all of science.

We now list some additional properties which can be proven from the field postulates. These properties are useful in helping us understand the operations and arithmetic of real numbers.

3.12. Rules of sign.

(a) $+(+a) = a$, also $+a = a$.

(b) $-(+a) = -a$, also $(-a) \cdot (b) = -a \cdot b$.

(c) $+(-a) = -a$, also $a \cdot (-b) = -a \cdot b$.

(d) $-(-a) = a$, also $(-a) \cdot (-b) = a \cdot b$.

(e) $(-a) + (-b) = -(a + b)$, also $a + b = +(a + b)$.

3.13. Equality laws. If a, b, c are real numbers with $a = b$ then

(a) $a + c = b + c$,

(b) $a \cdot c = b \cdot c$.

3.14 Cancellation laws. Let a, b, c be real numbers

(a) If $a + c = b + c$, then $a = b$.

(b) If $a \cdot c = b \cdot c$, $c \neq 0$ then $a = b$.

3.15. If $a \cdot b = 0$, then $a = 0$ or $b = 0$ or both.

In this lesson and in the next few lessons we shall be applying the above properties to practical problems involving the real numbers. Also many times whenever multiplication is clearly understood we will write $a \cdot b$ as ab or even (a)(b).

Let us first consider the rule of signs to use with grouping symbols. The usual grouping symbols used in mathematics are parentheses ( ), brackets [ ], and braces { }. A grouping symbol signifies to us that all of the elements or numbers inside the grouping symbol are to be considered as one element or number.

For example, there are two terms in the following expression:

$$-(13 - 6) + (-21 + 33),$$

and within each term there is again two terms. This expression can be simplified by using the rule of signs 3.12, (e) above in either of two ways.

The first of these ways involves removing the grouping symbols before combining like terms or numbers which can be thought of as the distributive property. Thus,

$$\begin{aligned} -(13 - 6) + (-21 + 33) &= -13 + 6 - 21 + 33 \\ &= 39 - 34 = 5. \end{aligned}$$

The second method involves combining like terms or numbers before the grouping symbols are removed. Thus,

$$\begin{aligned} -(13 - 6) + (-21 + 33) &= -(7) + (12) \\ &= -7 + 12 = 5. \end{aligned}$$

PROBLEM 3.16: Simplify the expression given below in each of two ways.

$$15 - \{9 - [(6 - 32) - (-25 - 8)]\} =$$

$$15 - 9 + [(6 - 32) - (-25 - 8)] =$$

$$15 - 9 + (6 - 32) - (-25 - 8) =$$

$$15 - 9 + 6 - 32 + 25 + 8 =$$

$$54 - 41 = 13.$$

Next we combine within the inner-most symbol, and then begin removing the grouping symbols according to 3.12 (a), (b), (c), and (d).

$$15 - \{9 - [(6 - 32) - (-25 - 8)]\} =$$

$$15 - \{9 - [(-26) - (-33)]\} =$$

$$15 - \{9 - [-26 + 33]\} =$$

$$15 - \{9 - [7]\} =$$

$$15 - \{9 - 7\} = 15 - \{2\} = 15 - 2 = 13.$$

The rule of signs (property 3.12) also holds for division of real numbers.

Thus,

$$+ \frac{+a}{+b} = \frac{a}{b} ,$$

$$- \frac{+a}{+b} = - \frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} ,$$

$$- \frac{-a}{+b} = \frac{+a}{+b} = \frac{-a}{-b} = \frac{a}{b} .$$

We see that there are several ways to write an algebraic expression with one negative sign.

PROBLEM 3.17: Simplify the following expression.

$$\frac{46 - 10}{-23 - (-3)} \cdot \frac{24 - (-13 - 11)}{2 - [3 - (6 - 1)]} .$$

We first combine numbers within the grouping symbols.

Thus,

$$\begin{aligned}
\frac{46-10}{-23-(-3)} \cdot \frac{24-(-13-11)}{2-[3-(6-1)]} &= \frac{46-10}{-23-(-3)} \cdot \frac{24-(-24)}{2-[3-(5)]} \\
&= \frac{46-10}{-23+3} \cdot \frac{24+24}{2-[3-5]} \\
&= \frac{36}{-20} \cdot \frac{48}{2-[-2]} = \frac{36}{-20} \cdot \frac{48}{4} \\
&= \frac{36}{-20} \cdot \frac{12}{1} = \frac{9}{-5} \cdot \frac{12}{1} \\
&= \frac{108}{-5} = -\frac{108}{5} .
\end{aligned}$$

The same rules of sign apply whenever we are using general or literal numbers. Most of the time the symbols for general numbers will be letters of the English or Greek alphabet. In using general numbers the distributive law (postulate 3.11) tells us how to add or subtract like terms. For example, consider the algebraic expression, $2a + 3a$ where a is a general number.

Then

$$2a + 3a = a \cdot 2 + a \cdot 3 = a \cdot (2 + 3) = a \cdot 5 = 5a.$$

PROBLEM 3.18: Simplify and combine like terms in the following algebraic expression.

$$6a - (2a - b) - [3b - (5a + 2b)]$$

$$\begin{aligned}
&6a - (2a - b) - [3b - (5a + 2b)] = \\
&6a - 2a + b - 3b + (5a + 2b) = \\
&4a - 2b + 5a + 2b = 9a.
\end{aligned}$$

Exercises for Lesson 3.

1. Prove the equation $(b + c)a = ab + ac$ from the field postulates.

2. In the following give the appropriate postulate(s) or property(ies) which allows us to make the following statements.

(a) $(-5)(-6) = 30$

(b) $x - (-y) = x + y$

(c) If $4p = 0$ then $p = 0$.

(d) If $\frac{a}{b} = q$, $b \neq 0$ then $a = bq$

(e) If $x + a = b$ then $x = b - a$

(f) If $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$ then $a = b$.

(g) $\dfrac{a}{\frac{b}{c}} = \dfrac{ac}{b}$, $(b,c \neq 0)$.

(h) If $6(x - y) = 3z$ then $2(x - y) = z$.

In problems 3 - 17, perform the indicated operations.

3. $(12.37) - (-7.45) - (9.23) + (-11.56)$

4. $(9 - 13) - (-7 + 12) + (17 - 11) - (-9 - 10)$.

5. $19 - \{9 - [2 - (13 - 28)]\}$.

6. $(11 - 13) \cdot (2 - 9) \cdot (17 - 12 - 25)$

7. $(13 - 15) \cdot (-2) - \{22 - [7 - (15 - 37)]\}$.

8. $[(37 - 19) - (35 - 14)] \cdot [(45 - 41) - (12 - 17)]$.

9. $\dfrac{39 - (-6)}{-8 - 7} \cdot \dfrac{25 - 53}{-19 + 15}$

10. $\dfrac{7\ (-6) + 78}{(-4)(-3) - (-6)} \cdot \dfrac{10 + 5\ (-3)}{4\ (-6) + 25}$

11. $\dfrac{(-4)(-5) + 8}{-11 + (-3)(-6)} \cdot \dfrac{(-4) \cdot (-6) \cdot (-5)}{(-3) \cdot (-40)}$

12. $8a - [(5a - 4) - (4a + 1)]$

13. $8y - (3x - 7y) - [4x - (3x - y)]$

14. $9s - \{6r - [(r - 4s) - (5r + 7s)]\}$.

15. $12a - \{10b - [(4b - 3a) - (6a - 5a)]\}$.

16. $[(d - 3 - 2f) - (3d + 2e + 5f)] - (5d + 4e - 3f)$

17. $9c - [(4c + 8) - (7c - 6)]$.

In problems 18 - 20 write an equivalent expression wherein the last three terms are enclosed in parentheses preceded by a minus sign.

18. $6 - 2x + y - 3$

19. $-a - 4b + 3c - 6d$

20. $-5 - r + 8s + 3t$.

In problems 21 - 24, find the additive inverse and multiplicative inverse of the following numbers.

21. $6/3$

22. π

23. $7 + 14 \div 7 + 2 \cdot 3 - 1$

24. $\left(\frac{3}{8} - \frac{7}{8}\right) \div \frac{1}{2}$

Answers for Lesson 3.

2. (a) Rules of sign 3.12 (d)
   (b) Rules of sign 3.12 (d)
   (c) 3.15
   (d) $\frac{a}{b} = q$
   $b \cdot \frac{a}{b} = b \cdot q$ Equality laws 3.13 (b)
   $b \cdot \left(\frac{1}{b} \cdot a\right) = b \cdot q$
   $\left(b \cdot \frac{1}{b}\right) \cdot a = b \cdot q$ Associative law of multiplication 3.8
   $1 \cdot a = b \cdot q$ Multiplicative inverse 3.10
   $a = b \cdot q$ Multiplicative identity 3.9
   (e) $x + a = b$
   $x + a + (-a) = b + (-a)$ Equality laws 3.13 (a)
   $x + 0 = b + (-a)$ Additive inverse 3.5
   $x = b + (-a)$ Additive identity 3.4
   $x = b - a$ Rules of sign 3.12 (c)

3. $-.97$
4. 16
5. 27
6. -280
7. 11
8. -27
9. -21
10. -10
11. -4
12. $7a + 5$
13. $14y - 4x$
14. $-10r - 2s$
15. $8a - 6b$
16. $-3 - 7d - 6e - 4f$
17. $12c - 14$
18. $6 -(2x - y + 3)$
19. $-a -(4b - 3c + 6d)$
20. $-5 -(r - 8s - 3t)$
21. $\frac{-6}{3}, \frac{3}{6}$
22. $-\pi, \frac{1}{\pi}$
23. $-14, \frac{1}{14}$
24. $+1, -1$

LESSON 4

Algebra

In lesson 3 of this series of algebra we learned the basic properties of the real number system and how to simplify the addition and subtraction of simple algebraic expressions. In this lesson we will learn about multiplication and division of algebraic expressions. In order to simplify multiplication and division in algebra we will use a shorthand notation called exponents. We begin with the following definition.

DEFINITION 4.1. If n is a counting number then $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$.

In the expression a^n the number n is called the *exponent of a and a is* sometimes called the base.

From this definition we see that

$$a^4 = a \cdot a \cdot a \cdot a$$

$$5^3 = 5 \cdot 5 \cdot 5$$

$$(ab)^2 = (ab) \cdot (ab)$$

Also from definition 4.1 several properties of exponents can be proven.

PROPERTY 4.2. If m and n are counting numbers then

$$a^m \cdot a^n = a^{m+n}.$$

Proof of property 4.2:

$$a^m = \underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}} \quad \text{and} \quad a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

By property 4.2,

$$a^m \cdot a^n = \underbrace{a\cdot a\cdot \cdots \cdot a}_{m}\cdot\underbrace{a\cdot a\cdot \cdots \cdot a}_{n} = a^{m+n}$$

since there are m+n factors of a.

This rule simply states that multiplication involving exponents acting on a common base can be accomplished through addition of the exponents.

Another property of exponents is the following:

PROPERTY 4.3. If n is a counting number then

$$(ab)^n = a^n b^n.$$

The proof is left as an exercise for the student.

We now give some examples of properties 4.2 and 4.3.

$$x^3 \cdot x^5 \cdot x = x^8 \cdot x = x^9$$

$$(3y)^4 = 3^4 y^4 = 81y^4$$

$$\begin{aligned}(-2a)^2(3a)^3 &= (-2)^2a^2 \cdot 3^3a^3\\ &= 4a^2 \cdot 27a^3\\ &= 108a^5\end{aligned}$$

PROPERTY 4.4. If m and n are counting numbers then

$$(a^m)^n = a^{mn}.$$

The proof is left as an exercise for the student.

From property 4.4 we have

$$(5^2)^3 = 5^6 = 15,625.$$

Also applying properties 4.3 and 4.4 we have

$$\begin{aligned}(-a^3b^2)^5 &= (-1)^5(a^3)^5(b^2)^5\\ &= -a^{15}b^{10}.\end{aligned}$$

The next property deals with division of like bases.

PROPERTY 4.5. If m and n are counting numbers then

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{, m greater than n} \\ 1 & \text{, m = n} \\ \dfrac{1}{a^{n-m}} & \text{, m less than n.} \end{cases}$$

Thus, in dividing numbers with like bases the rule is to subtract exponents as given by property 4.5.

Examples are:

$$\frac{10^{13}}{10^7} = 10^{13-7} = 10^6$$

$$\frac{b^3}{b^3} = 1$$

$$\frac{12a^3b^5}{6a^7b^2} = \frac{2b^{5-2}}{a^{7-3}} = \frac{2b^3}{a^4}\ .$$

One of the most prevalent problems of exponents is failure to recognize the base. For example, $3x^2 \neq (3x)^2$ since in the first case the base is x and in the second case the base is 3x.

Thus,

$$-3^2 = -(3 \cdot 3) = -9$$

while

$$(-3)^2 = (-3)(-3) = 9.$$

PROBLEM 4.6. Simplify by applying the laws of exponents.

$$\left(\frac{48x^2y^5}{-32x^2y}\right)^3$$

$$\left(\frac{48x^2y^5}{-32x^2y}\right)^3 = \left(\frac{3(16)}{-2(16)} \cdot \frac{x^2}{x^2} \cdot \frac{y^5}{y}\right)^3$$

$$= \left(\frac{3}{-2} \cdot 1 \cdot y^4\right)^3 = \left(\frac{3y^4}{-2}\right)^3$$

$$= \frac{3^3y^{12}}{(-2)^3} = \frac{27y^{12}}{-8} = \frac{27}{-8}\,y^{12}.$$

With the use of the distributive law (postulate 3.11) and the properties of exponents we are now in a position to multiply and divide algebraic expressions.

First, consider

$$5x^2y(3x^2 + 2xy - y^2).$$

Then by the distributive law this is

$$(5x^2y)(3x^2) + (5x^2y)(2xy) - (5x^2y)y^2.$$

By the rules of exponents this becomes

$$15x^4y + 10x^3y^2 - 5x^2y^3.$$

As another example consider,

$$(2a + 5)(3a^2 - a + 4).$$

Thinking of 2a + 5 as one quantity, since addition and multiplication are closed, we have by the distributive law

$$(2a + 5)(3a^2) - (2a + 5)(a) + (2a + 5)(4).$$

Again using the distributive law we have

$$(2a)(3a^2) + 5(3a^2) - (2a)(a) - (5)(a) + (2a)(4) + (5)(4),$$

and using the properties of exponents we can write this as

$$6a^3 + 15a^2 - 2a^2 - 5a + 8a + 20.$$

Adding like terms we then have

$$6a^3 + 13a^2 + 3a + 20.$$

PROBLEM 4.7. Multiply the expressions and collect like terms.

$$(3x^2 + 2x - 5)(2x^2 - x + 2)$$

$$\begin{aligned}(3x^2 + 2x - 5)(2x^2 - x + 2) &= (3x^2 + 2x - 5)(2x^2)\\ &\quad -(3x^2 + 2x - 5)(x)\\ &\quad +(3x^2 + 2x - 5)(2)\\ &= 6x^4 + 4x^3 - 10x^2\\ &\quad -3x^3 - 2x^2 + 5x\\ &\quad +6x^2 + 4x - 10\end{aligned}$$

Collecting like terms we have

$$(3x^2 + 2x - 5)(2x^2 - x + 2) = 6x^4 + x^3 - 6x^2 + 9x - 10.$$

Division of algebraic expressions can be broken down into two cases. In the first case we consider division by a single termed expression such as

$$\frac{8x^4 - 12x^2 + 9}{-4x^2}.$$

From addition and subtraction of rationals this can be thought of as

$$\frac{8x^4}{-4x^2} - \frac{12x^2}{-4x^2} + \frac{9}{-4x^2}$$

$$= -2x^2 + 3 - \frac{9}{4x^2}.$$

As another example consider

$$\frac{6a^3b^2 - 4ab^2 + 12a^4b^5}{2ab^2}.$$

Then this is equal to

$$\frac{6a^3b^2}{2ab^2} - \frac{4ab^2}{2ab^2} + \frac{12a^4b^5}{2ab^2}$$

$$= 3a^2 - 2 + 6a^3b^3.$$

In the second case for division we consider division by a many termed expression such as

$$\frac{6a^2 - a - 15}{2a + 3}$$

Even though it is true that

$$\frac{6a^2 - a - 15}{2a + 3} = \frac{6a^2}{2a + 3} - \frac{a}{2a + 3} - \frac{15}{2a + 3},$$

this does not help simplify the expression.

A long division process is needed here similiar to the usual method of division of real numbers except we use the properties of exponents.

$$\begin{array}{rl} & 3a - 5 \\ 2a + 3 & \overline{)\,6a^2 - a - 15} \\ & \underline{6a^2 + 9a} \\ & \quad -10a - 15 \\ & \quad \underline{-10a - 15} \\ & \qquad\qquad 0 \end{array}$$

1. Multiply (2a + 3) by 3a to give $6a^2 + 9a$.
2. Subtract to give -10a
3. Bring down -15
4. Multiply (2a + 3) by (-5) to give -10a - 15.
5. Subtract to find remainder.

Thus we see that

$$\frac{6a^2 - a - 15}{2a + 3} = (3a - 5) + \frac{0}{2a + 3}$$

The algebraic expression 3a - 5 is called the quotient of $\frac{6a^2 - a - 15}{2a + 3}$ while 0 is called the remainder. Also, 2a + 3 is called the divisor while $6a^2 - a - 15$ is called the dividend. In any division process we have the following:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}} .$$

PROBLEM 4.8. Divide the expression $\frac{3x^2 + 2x - 1}{x - 2}$ and write your answer in the form

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}} .$$

$$\begin{array}{rl} & 3x + 8 \\ x - 2 & \overline{)\,3x^2 + 2x - 1} \\ & \underline{3x^2 - 6x} \\ & \qquad 8x - 1 \\ & \qquad \underline{8x - 16} \\ & \qquad\quad 15 \end{array}$$

Hence,

$$\frac{3x^2 + 2x - 1}{x - 2} = (3x + 8) + \frac{15}{x - 2} .$$

Note: In most division problems involving algebraic expressions there may be values of the variable which make the division undefined. For example in problem 4.8 if $x = 2$ then the division is not possible because the divisor is zero. Thus, it is assumed that $x \neq 2$ in order to do the division.

Exercises for Lesson 4

1. Prove the following properties of exponents. m and n are counting numbers.

(a) $(ab)^m = a^m b^m$

(b) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

(c) $(a^m)^n = a^{mn}$

In problems 2 - 15 simplify by performing indicated operations.

2. $(-3)^2(-3)^3$

3. $-(-7)^2(-1)^5$

4. $(t^4)^7$

5. $(-3x^2y)^3$

6. x[illegible]$\cdot x^{2m-1}$

7. $(xy^2)^2(xy)^5$

8. $5u^2v^3 \cdot 4uv^5 \cdot 6u^4v^2$

9. $\frac{-x^7}{x^{11}}$

10. $\frac{6c^{13}d^9}{45c^{10}d^4}$

11. $\frac{108u^8v^7}{60u^{12}v^4}$

12. $\left(\frac{a^9}{a^9}\right)^3$

13. $\left(\frac{-8b^5}{b^8}\right)^4$

14. $\left(\frac{35a^5b^4c}{21a^2bc^6}\right)^3$

15. $\left(\frac{6a^4b^5c}{12a^7b^3d}\right)^6$

In problems 16 - 23 perform the indicated multiplication and collect like terms.

16. $-4x^4y^2(x^4-2x^2y + 8y + 3)$

17. $(x + 2)(x - 3)$

18. $(2a - 3)^2$

19. $(8u^3 + 7)(2u^3 - 9)$

20. $(a+b)(a^2 - ab + b^2)$

21. $(4 - x)(8 - 3x)(1 + 2x)$

22. $(a + b + c)^2$

23. $(x^2 + 4xy - 6y^2)(2x^2 - xy - y^2)$

In problems 24 - 30 perform the indicated divisions and write as

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}} .$$

24. $\dfrac{8x^2 + 12v^2}{4x}$

25. $\dfrac{30a^5 - 10a^4 + 18a^3 - 9}{15a^4}$

26. $\dfrac{125x^3 - 8y^3}{5x - 2y}$

27. $\dfrac{3m^3 + 8m - 5}{m - 6}$

28. $\dfrac{5 + 6v - 29v^2 - 6v^3}{5 + v}$

29. $\dfrac{x^3 - 3x + 2}{x + 2}$

30. $\dfrac{x^8 - 16y^{16}}{x^2 - 2y^4}$

In problems 31 - 34 evaluate by substituting the numerical values for the variables and simplify the result. Let $x = 2$, $y = 3$, and $z = -3$.

31. $2x^2 - 6x + 3$

32. $\dfrac{2x^2 - xy}{x^2 + 4xy}$

33. $\dfrac{3x^2 - 6xy^2 - z}{(xy)^2 - z^2}$

34. $-(-x)^2(-y)^2(-z)^3$.

Answers for Lesson 4.

2. -243
3. 49
4. t^{28}
5. $-27x^6y^3$
6. x^{3m-1}
7. x^7y^9

10. $\frac{2c^3d^5}{15}$

11. $\frac{9v^3}{5u^4}$

12. 1

13. $\frac{4096}{b^{12}}$

14. $\frac{125a^9b^9}{27c^{15}}$

15. $\frac{b^{12}c^6}{64a^{18}d^6}$

16. $-4x^8y^2 + 8x^6y^3 - 32x^4y^3 - 12x^4y^2$
17. $x^2 - x - 6$
18. $4a^2 - 12a + 9$
19. $16u^6 - 58u^3 - 63$
20. $a^3 + b^3$
21. $32 + 44x - 37x^2 + 6x^3$
22. $a^2 + 2ab + 2ac + 2bc + b^2 + c^2$
23. $2x^4 + 7x^3y - 17x^2y^2 + 2xy^3 + 6y^4$
24. $2x + \frac{3v^2}{x}$
25. $2a - \frac{2}{3} + \frac{6}{5a} - \frac{3}{5a^4}$
26. $25x^2 + 10xy + 4y^2$
27. $3m^2 + 18m + 116 + \frac{+691}{m-6}$
28. $1 + v - 6v^2 + \frac{0}{5 + v}$
29. $x^2 - 2x + 1 + \frac{0}{x + 2}$
30. $x^6 + 2x^4y^4 + 4x^2y^8 + 8y^{12} + \frac{0}{x^2 - 2y^4}$

31. -1
32. $\frac{1}{14}$
33. $\frac{-31}{9}$
34. 972

LESSON 5

Algebra

One reason algebra is so universally important, is the fact it uses symbols rather than the usual numerals to stand for numbers. By agreeing on certain shortcuts such as exponents for example, a few algebraic symbols can take the place of a great many words.

For example, what physicist in trying to teach a student about the distance an object freely falling from rest would say,

> The distance, s in feet, through which a body falls from rest in t seconds, is equal to half the product of the acceleration g (32 ft. per sec^2.), caused by the pull of the earth and the square of the time t.

How much simpler this can be said by using the algebraic expression:

$$s = \frac{1}{2} gt^2.$$

The relationship we have just expressed relates two quantities; namely the distance s in feet to the time t in seconds, through which the object has fallen. Any statement of equality between two quantities is called an equation or formula. Equations or formulas occur so frequently in our lives that a thorough knowledge of what operations can be performed on equations and how to solve equations for one quantity in terms of another quantity is very important.

For example, suppose we wish to buy a used car and pay off the loan in one year. We can afford a total loan of not more than $1,000, (that is, the amount borrowed P plus interest I must not exceed $1,000). The bank has agreed to loan the money at $r = 8\%$ interest. What is the maximum amount P we can pay for a car? The banker uses the following formula for a one year loan:

$$A = P(1 + r),$$

where A is the total amount, P is the amount borrowed (the principle) and r is the interest. Thus to solve this problem we let A = \$1,000 and $r = .08$. Thus we have

$$1000 = P(1 + .08)$$

$$1000 = P(1.08)$$

and

$$P = \frac{1000}{1.08} = \$925.93.$$

Therefore, the maximum amount we can pay for the car is approximately \$926.

In solving equations there are certain rules and operations which must be followed. These are basically given as the equality laws (property 3.13) and the cancellation laws (property 3.14) of lesson 3.

The equality laws state that equals can be added or multiplied by equals and the result will be equal.

For example, if to each side of the equation $9 = 9$ we add 3, the new equation $12 = 12$ is obtained. Or if we multiply each side of the equation $9 = 9$ by 4, we have the new equation $36 = 36$. These examples are not very useful even though they do illustrate the equality laws. These laws do become useful however, when we want to solve an equation for a variable. Consider the equation

$$x - 5 = 14.$$

If we add 5 to both sides of the equation, we have

$$x - 5 + 5 = 14 + 5,$$

or

$$x = 19.$$

Substitution of $x = 19$ into the original equation provides a way of checking the equation to be sure this is the correct answer.

Next, consider the equation

$$\frac{x}{3} = 4.$$

Multiplying both sides of the equation by 3 will result in

$$3\left(\frac{x}{3}\right) = 3 \cdot 4$$

or

$$x = 12.$$

Again checking, we substitute $x = 12$ into the equation and see if equality is obtained.

$$\frac{12}{3} = 4$$

or

$$4 = 4.$$

Thus $x = 12$ is the correct answer.

The cancellation laws state that equals can be subtracted or divided by equals (zero excepted) and the result will be equal.

For example, to solve the equation

$$2x + 5 = 3$$

we first subtract 5 from both sides and obtain

$$2x + 5 - 5 = 3 - 5$$

or

$$2x = -2.$$

Next, we divide both sides by 2 and obtain

$$\frac{2x}{2} = \frac{-2}{2}$$

or

$$x = -1.$$

Now, checking to be sure we have the right answer we substitute $x = -1$ into the original equation.

$$2(-1) + 5 = 3$$
$$-2 + 5 = 3$$
$$3 = 3.$$

Since equality is obtained, $x = -1$ is the correct answer.

We now give some examples of solving equations where just one variable is involved. If that variable is x, then the principal idea is to reduce the equation to the form $ax = b$ by employing addition, subtraction, or multiplication. Once we have the equation reduced to the form $ax = b$, then it can always be solved for x by dividing by a ($a \neq 0$) or by multiplying by $\frac{1}{a}$.

PROBLEM 5.1: Solve and check.

$9x - 4 = 10x - 6$ or $9x - 4 = 10x - 6$

(+4) $9x = 10x - 2$ (+6) $9x + 2 = 10x$

(-10x) $-x = -2$ (-9x) $2 = x$

multiply (-1) $x = 2.$

Check: If $x = 2$ then

$$9(2) - 4 = 10(2) - 6$$
$$18 - 4 = 20 - 6$$
$$14 = 14.$$

PROBLEM 5.3: Solve and check.

$$9(2y + 7) = 5(4y + 11)$$

By removing parentheses we obtain

	$18y + 63 = 20y + 55$	or	$18y + 63 = 20y + 55$
(-63)	$18y = 20y - 8$	(-55)	$18y + 8 = 20y$
(-20y)	$-2y = -8$	(-18y)	$8 = 2y$
divide (-2)	$y = 4.$	divide (2)	$4 = y.$

Check: If $y = 4$ then

$$9(2(4) + 7) = 5(4(4) + 11)$$

$$9(8 + 7) = 5(16 + 11)$$

$$9(15) = 5(27)$$

$$135 = 135.$$

PROBLEM 5.3: Solve and check.

$$\frac{4x - 23}{6} + \frac{1}{3} = \frac{5x}{4}$$

Multiplying both sides by 12 will eliminate fractions from the equation.

Thus, we have

$$2(4x - 23) + 4 = 3(5x)$$

or $\quad 8x - 46 + 4 = 15x$

or $\quad 8x - 42 = 15x$

(+42) $\quad 8x = 15x + 42$

(-15x) $\quad -7x = 42$

divide (-7) $\quad x = -6$

Check: $\quad \dfrac{4(-6) - 23}{6} + \dfrac{1}{3} = \dfrac{5(-6)}{4}$

$$\frac{-24 - 23}{6} + \frac{1}{3} = \frac{-30}{4}$$

$$\frac{-47}{6} + \frac{1}{3} = \frac{-15}{2}$$

$$\frac{-47}{6} + \frac{2}{6} = \frac{-45}{6}$$

$$\frac{-45}{6} = \frac{-45}{6}$$

Exercises for Lesson 5.

Solve and check each of the following equations for the letter involved:

1. $10x + 7 = 15x - 8$
2. $7(5y + 13) = 5(3y + 7)$
3. $\frac{1}{4} a - \frac{7}{12} = \frac{11}{12} - \frac{5}{4} a$
4. $2x(x + 1) - 4 = x(2x + 5) + 6$
5. $\frac{17}{4} y + 8 = \frac{13}{4} y + 15$
6. $3w(w - 1) = 4w(w + 2) - (w^2 + 12w - 16)$
7. $4s(3s - 1) + 19 = 2s(6s + 5) - 9$
8. $7 + 6v(4v + 3) = 3v(8v + 5) - 20$
9. $(2x + 3)(5x - 1) = 10x^2 + 3x - 4$
10. $(4y + 5)(2y + 7) = (y + 2)(6y + 20) + (2y - 1)(y + 3)$
11. $\frac{2m - 17}{3} = \frac{8m + 11}{6}$
12. $\frac{7y - 1}{9} - \frac{3y + 4}{8} = 1$
13. $\frac{8k + 3}{6} + \frac{1}{2} = \frac{7k - 1}{3}$
14. $\frac{6v - 5}{7} = \frac{4v - 7}{3}$
15. $(5 - 3c)(7 + 4c) + (3 - 2c)(11 - 6c) = -14.$

Answers for Lesson 5.

1. $x = 3$
2. $y = \frac{-14}{5}$
3. $a = 1$
4. $x = \frac{-10}{3}$
5. $y = 7$
6. $w = 16$
7. $s = 2$
8. $v = -9$
9. $x = -\frac{1}{10}$
10. $y = 2$
11. $m = \frac{-45}{4}$
12. $y = 4$
13. $k = \frac{4}{3}$
14. $v = \frac{17}{5}$
15. $c = 2$

LESSON 6

Algebra

From lesson 5 we learned to solve equations with one variable. Using the same rules that applied to those equations, we can also solve equations with more than one variable. These equations appear to be more complex because many algebraic operations cannot be fully carried out, but must be left in indicated form. For example, when told to add x and y we write this in indicated form as $x + y$, but when told to add 3x and 10x we write as 13x. In the first situation the addition is indicated, but cannot be done until we know the value of x and y. In the second case the addition is possible since $3x + 10x = (3 + 10)x = 13x$.

There are many practical problems in mathematics, science, and business which require the solution of a formula for one of the variables involved. For example, in lesson 5 we considered the equation

$$A = P(1 + r),$$

where we solved for P after we knew A and r. We could have just as well solved for P without having a value for A and r by dividing by $1 + r$. Thus,

$$\frac{A}{1 + r} = P.$$

Then if we are given any value of A and r we can substitute into the equation and find P. This formula which expresses P in terms of A and r is quite useful to the banker. On the other hand, before Congress passed the truth in lending law, the equation solved for r might be very useful to a person making a loan. He would know the interest he is being charged, since r would be expressed in terms of the total amount of the loan A and the principal P. Thus, solving for r we obtain

$$A = P(1 + r)$$

$$A = P + Pr$$

$$A - P = Pr$$

$$r = \frac{A - P}{P} .$$

We will now solve some problems whose equations involve more than one variable.

PROBLEM 6.1: Solve for a.

$$5ab - b^2 = 9b^2.$$

$(+b^2)$ $$5ab = 10b^2$$

Divide $(5b)$ $$a = \frac{10b^2}{5b} = 2b.$$

We note here, that in solving an equation with more than one variable, we combine all terms involving the variable for which we want to solve for on one side of the equation, and take all other terms to the other side of the equation. This point is illustrated in the next few problems.

PROBLEM 6.2: Solve for c.

$$7(a^2b + c) - 4a^2(b + 3a^2) = 10c.$$

We first remove parentheses,

$$7a^2b + 7c - 4a^2b - 12a^4 = 10c.$$

Since we are solving for c we want all the terms involving c on one side. Thus subtracting 7c from both sides we obtain

$$7a^2b - 4a^2b - 12a^4 = 3c$$

or

$$3a^2b - 12a^4 = 3c.$$

Dividing by 3 we have

$$c = a^2b - 4a^4.$$

PROBLEM 6.3: Solve for r.

$$\frac{e^2r + 8}{2k} = \frac{e^2r + 10}{3k}.$$

Multiply (6k) $3(e^2r + 8) = 2(e^2r + 10)$

or $3e^2r + 24 = 2e^2r + 20.$

Collecting r terms gives $e^2r + 24 = 20$

(-24) $e^2r = -4$

Divide (e^2) $r = \frac{-4}{e^2}.$

PROBLEM 6.4: Solve for y.

$$\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$$

Multiplying by xyz will eliminate fractions in this equation.

Thus,

$$yz = xz + xy.$$

Now collecting y terms by subtracting xy from both sides we have

$$yz - xy = xz.$$

Using the distributive law we have

$$y(z - x) = xz$$

Dividing by z - x yields

$$y = \frac{xz}{z - x}.$$

There is one other type of problem in science which is similar to the ones we have studied, except that the problem is stated in words instead of equation form. These are known as word or statement problems, and there is one main difference in solving these problems. They must first be converted from English words to an equation, then solved by the methods of this lesson.

PROBLEM 6.5: The sum of a number and one-fourth its value is 20. Find the number. Let x be the symbol for the number. Then according to the problem

$$x + \frac{1}{4}x = 20$$

or

$$\frac{5}{4}x = 20.$$

Multiplying both sides by 4/5 yields

$$x = \frac{4}{5}\cdot 20 = 16.$$

PROBLEM 6.6: Joe and Jim together earned \$72. Both were paid at the same rate, but Jim worked three times as long as Joe. How much did each receive? Let x be the number of dollars Joe received, and 3x, the number of dollars Jim received. Thus,

$$x + 3x = 72$$

or

$$4x = 72$$

and

$$x = \$18.$$

Hence, Joe received \$18, while Jim received 3(18) or \$54.

Exercises for Lesson 6.

Solve the following equations for the indicated variables.

1. $3n^2v + 10n^4 = 34n^4$; v
2. $4x - 3y - (10x + 7y) = 0$; x
3. $5(4r - 3h) - 2(7r - 9h) = 3r$; r
4. $4(x + ky) = k^4 + 5x$; y
5. $\frac{x}{a^4} - \frac{3}{2a^4} = 2$; a^4
6. $\frac{2x + v}{4} - \frac{6x + 3v}{7} = \frac{15v}{28}$; v.

7. $\frac{m^2t - 3}{4p} = \frac{2m^2t - 9}{5p}$; t, Can you solve for p?

8. $(hy + 7)(hy - 3) = hy(hy + 1)$; h, y.

9. $V = \frac{1}{3} \pi r^2 h$; h

10. $C = \frac{5}{9} (F - 32)$; F

11. $F = \frac{w}{g} a$; a

12. $l = a + (n - 1)d$; a, n, d.

13. $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$; u, v, f.

14. $I = \frac{f}{R + nr}$; R, n, r.

15. $A = P(1 + rt)$; t, r.

Solve the following word problems.

16. When x is multiplied by 7 the result is 56. What is x?

17. The sum of three whole numbers is 200. The smallest is 25 less than the largest. The largest is the largest by 3. What are the numbers?

18. The sum of the angles of a triangle is 180° . If in a certain triangle one angle is equal to 57° and the other two angles are equal, find the measure of the other angles.

19. Tom is three times as old as Sue. In 5 years he will only be twice as old as she is. How old are they?

20. Bob paid $215.39 more in federal income tax than he did in state income tax. He paid a total of $450.19. How much state tax did he pay?

Write the formula which corresponds to the following word statements.

21. The square of the diagonal d of a rectangle is equal to the sum of the squares of the two sides a and b.

22. The number F of degrees Fahrenheit that corresponds to a number C of degrees Centigrade is nine-fifths of C plus 32.

23. The current I in amperes flowing through a resistance R is equal to the voltage E in volts divided by the resistance.

24. The kinetic energy E of a moving body is equal to one-half the product of the mass m and the square of the velocity v.

25. The density d of a solid is equal to the mass m divided by the volume of the solid v.

Answers for Lesson 6.

1. $v = 8n^2$
2. $x = -\frac{5y}{3}$
3. $r = -h$
4. $y = \frac{k^4 + x}{4k}$
5. $a^4 = \frac{x}{2} - \frac{3}{4}$
6. $v = \frac{-x}{2}$
7. $t = \frac{7}{m^2}$, $p \neq 0$
8. $h = \frac{7}{y}$, $y = \frac{7}{h}$
9. $h = \frac{3V}{\pi r^2}$
10. $F = \frac{9C}{5} + 32$
11. $a = \frac{Fg}{w}$
12. $a = 1 - (n - 1)d$, $d = \frac{1 - a}{n - 1}$, $n = \frac{1 - a}{d} + 1$
13. $u = \frac{vf}{v - f}$, $v = \frac{uf}{u - f}$, $f = \frac{uv}{v + u}$
14. $R = \frac{f}{I} - nr$, $n = \frac{f}{Ir} - \frac{R}{r}$, $f = I(R + nr)$
15. $t = \frac{A - P}{Pr}$, $r = \frac{A - P}{Pt}$.
16. $x = 8$
17. 51, 73, 76
18. 46.5°
19. Tom is 15 years old.
    Sue is 5 years old.
20. \$117.40
21. $d^2 = a^2 + b^2$
22. $F = \frac{9}{5}C + 32$
23. $I = \frac{E}{R}$
24. $E = \frac{1}{2}mv^2$
25. $d = \frac{m}{V}$

SECTION III

Geometry

LESSON 7

Geometry

The birth of geometry was ushered in at the same time that many changes were taking place in world civilizations. The ancient civilizations of India, Babylonia, and Egypt were losing their influence while new peoples in the Mediterranean area, and especially the Greeks, were coming to the front. The Iron Age was beginning, which brought with it new changes in warfare and all pursuits requiring tools. The alphabet was invented and coins introduced. Trade was increasing and more and more discoveries were being made in all areas. Thus, the static outlook of the ancient orient, which had tried to answer the question of how, became impossible in a developing atmosphere of rationalism where men not only wanted to know how, but also why. For example, it was known by many ancient peoples that the base angles of an isosceles triangle are equal, but for the first time in mathematics men asked the fundamental question of "why are they equal?" The empirical processes of the ancient orient, quite sufficient for the question how, no longer sufficed to answer these more scientific inquiries of why. Thus, mathematics in the modern sense was born in this type of atmosphere, probably in one of the new trading towns along the west coast of Asia Minor. Tradition has it that demonstrative geometry began with Thales of Miletus during the first half of the sixth century B.C. Thales is one of the "seven wise men" of antiquity. He is the first known individual with whom mathematical discoveries are associated. Some results he is credited with are the following:

1. A circle is bisected by any diameter.
2. The vertical angles formed by two intersecting lines are equal.
3. An angle inscribed in a semicircle is a right angle.

The value of these results is not to be measured by the results themselves, since many were known long before Thales appeared on the scene, but in the fact that Thales supported them by some form of logical reasoning rather than intuition and experiment.

Even though Thales is credited with the beginnings of proof in geometry, he is almost overshadowed by Pythagoras, who was born about 572 B.C., on the Island of Samos. Pythagoras lived very close to and may have been a student of Thales early in his life. Pythagoras developed a school in southern Italy devoted to the study of philosophy, mathematics, and natural science. The school, known as the Pythagorean school, developed into a closely knit brotherhood with secret rites and observances. The philosophy of the school rested on the assumption that the whole numbers are the cause of the various qualities of matter. Thus, the study of numbers and number properties became an important topic in this school. All of the teaching was oral and all discoveries were credited to Pythagoras, so it is difficult to know what mathematical findings should be credited to Pythagoras himself, and which to other members of the fraternity. Tradition does, however, seem to be unanimous in ascribing to Pythagoras the discovery of the proof of the theorem on right triangles which bears his name. Even though the results of this theorem were known more than a thousand years earlier by the Babylonians, Pythagoras may have been the first to give a general proof of the theorem. We now state this important result.

THE PYTHAGOREAN THEOREM 7.1. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two legs.

In algebraic notation using figure 7.1

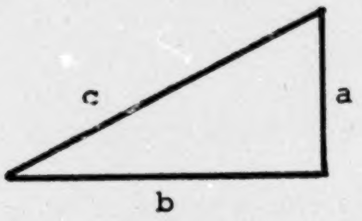


Fig. 7.1

we have that

$$c^2 = a^2 + b^2.$$

There have been many conjectures as to how Pythagoras discovered a proof for this theorem, but is is generally felt that it was some type of construction proof like the following.

Consider the triangle used in figure 7.1 above and construct two congruent squares each of sides a + b.

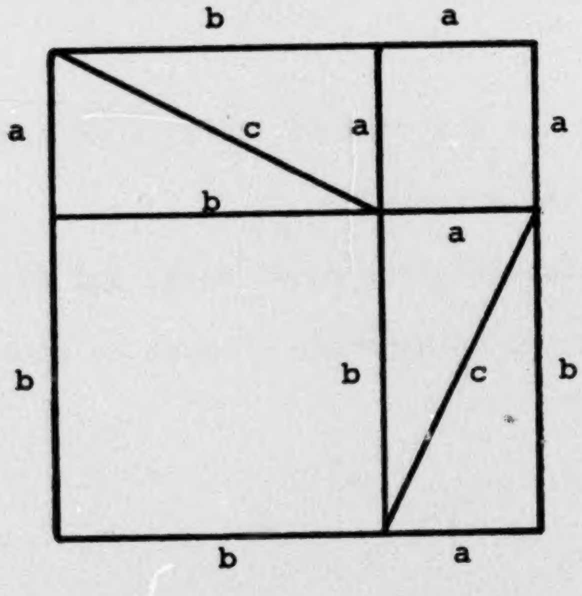


Figure 7.2

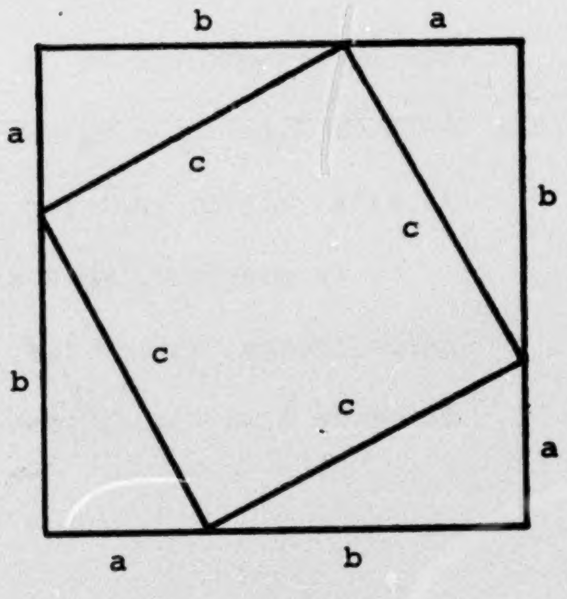


Figure 7.3

In figure 7.2 divide the square into six parts as shown. In figure 7.3 divide the square into five parts as shown. Now, certainly, the area in the total square of figure 7.2 is equal to

$$(a + b)^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

Now the area of the square in figure 7.3 is also equal to $(a + b)^2$ or can be thought of as the area of the square c which is c^2 plus the area of the four triangles each of area $\frac{1}{2}$ ab.

Thus,

$$(a + b)^2 = c^2 + 4(\frac{1}{2} ab) = c^2 + 2ab.$$

Hence, the area in the square of figure 7.2 is equal to the area in the square of figure 7.3 which implies that

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

or

$$a^2 + b^2 = c^2,$$

and the Pythagorean Theorem is proved.

Many different proofs of the Pythagorean Theorem have been given and we will consider several in the exercises.

Next let us consider some problems where the Pythagorean Theorem can be of some use to us.

PROBLEM 7.2. An airplane flies 12 miles due west of the airport and then 18 miles north. How far is it from the airport?

By constructing a right triangle with legs equal to 12 and 18 respectively, figure 7.4, we can use the Pythagorean Theorem to find the distance d to the airport.

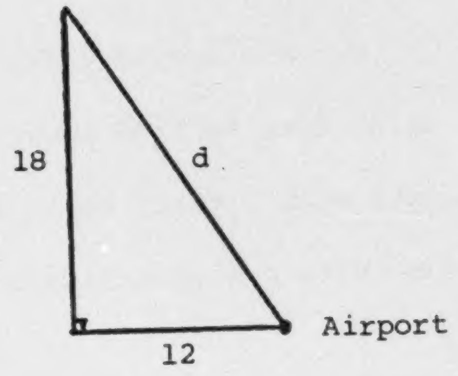

18
d
Airport
12

Figure 7.4

Thus,

$$d^2 = 18^2 + 12^2 = 324 + 144 = 468,$$

and

$$d \approx 21.6 \text{ miles}.$$

The symbol $\approx$ means approximately equal to.

As another example consider problem 7.3.

PROBLEM 7.3. A softball diamond is square in shape (figure 7.5). The bases are 60 feet apart. A catcher throws a ball to second base from home plate. How far did he throw the ball?

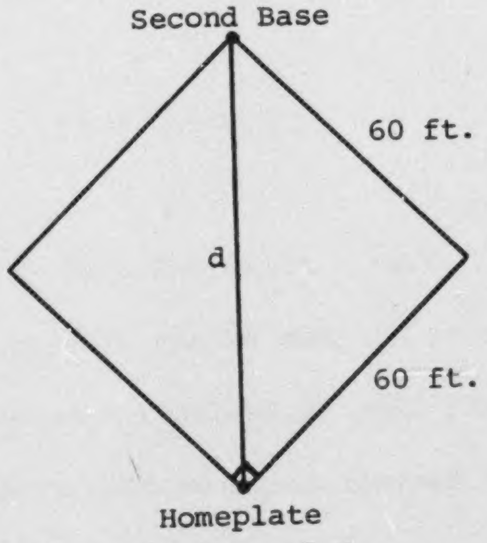

Second Base
60 ft.
d
60 ft.
Homeplate

Figure 7.5

By the Pythagorean Theorem,

$$d^2 = 60^2 + 60^2 = 3600 + 3600 = 7200$$

Thus,

$$d = \sqrt{7200} \approx 84.8 \text{ feet.}$$

The procedure we are using here to find square roots is a simple one and can be quite accurate when applied again and again. We shall illustrate the procedure with a simple example.

Suppose

$$x^2 = 11.$$

Then

$$x = \sqrt{11}, \text{ and we want to find}$$

$$x \text{ so that } x^2 = 11.$$

Now we know that

$$3^2 = 9$$

$$x^2 = 11$$

$$4^2 = 16.$$

Thus x is some number between 3 and 4. Also 11 is closer to 9 than to 16 so we believe x is closer to 3 than to 4. So we might guess that

$$x = 3.3.$$

Now

$$3.3^2 = 10.89$$

while

$$3.2^2 = 10.24 \text{ and } 3.4^2 = 11.56.$$

Thus 3.3^2 is closer to 11 than either 3.2^2 or 3.4^2, and if one decimal place accuracy is all that is desired, then we say that $x \approx 3.3$. Now if we want a second decimal place we then guess a number x between 3.3 and 3.4 such that

$$3.3^2 = 10.89$$

$$x^2 = 11$$

$$3.4^2 = 11.56.$$

Since 10.89 is closer to 11 than 11.56 we might guess the number x = 3.33. Now $3.33^2 = 11.0889$ and checking out the decimals just below and above 3.33 we have

$$3.32^2 = 11.0224$$
$$3.33^2 = 11.0889$$
$$3.34^2 = 11.1556.$$

From these calculations we see that 3.32^2 is closer to 11 than 3.33. Hence, we should also check 3.31^2 which is 10.9561. Now we see that 3.32^2 is closest to 11 of the numbers checked. Thus, for two decimal place accuracy we have

$$x = \sqrt{11} \approx 3.32.$$

This procedure can be continued to give any desired number of decimal place accuracy.

Exercises for Lesson 7.

1. Given a right triangle with sides a, b, and hypotenuse c (figure 7.1) find the missing side with one decimal place accuracy if

   (a) a = 3, b = 4, c = ?

   (b) a = 24, b = 17, c = ?

   (c) b = 8, c = 17, a = ?

   (d) b = 3, c = 13, a = ?

   (e) a = 40, c = 41, b = ?

2. If a diagonal of a square is 10 inches, how long is each side of the square?

3. A man walks 8 miles due east and then 5 miles due south. How far is he from his starting point?

4. A telephone pole is steadied by three guy wires. Each wire is to be fastened to the pole at a point 18 feet above the ground and anchored

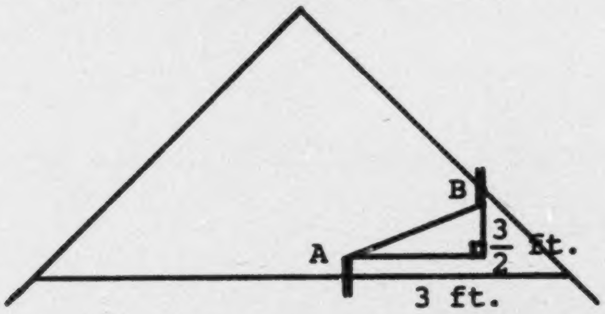
to the ground 10 feet from the base of the pole. How many feet of wire are needed for the three guy wires?

5. A cube has length 3 inches on a side. Find the length of the diagonal.

6. A gate is 4 feet wide and 6 feet high. How long is the brace that extends from one corner to the opposite corner?

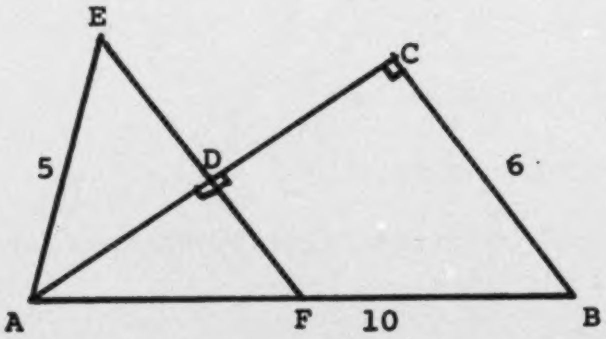
7. If the diagonal of a rectangle is 25 units and the length is twice the width, what are the dimensions of the rectangle?

8. A plumber, venting a hot water tank, finds that the vent in the roof does not line up with the vent in the ceiling.

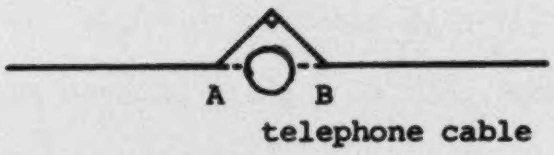


Find the length AB of vent pipe to be used if he must cut off 2 inches on each end of AB for the coupling joint.

9. In right triangle ABC, AB = 10, CB = 6. EF is perpendicular to and bisects AC. AE = 5. Find ED.



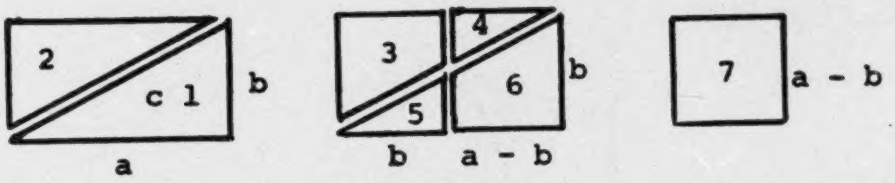
10. A gas line is being laid which must go over a telephone cable.



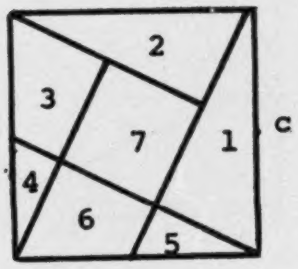
A dog-leg is made in the form of an isosceles right triangle. Find the length of each side of the dog-leg, if the length from A to B is 2 feet.

11. Show by the following demonstration that the Pythagorean Theorem is intuitively true.

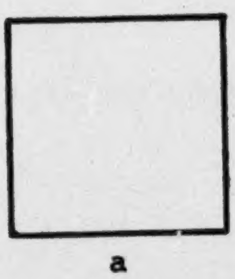
(a) Prepare seven pieces of cardboard with the measurements shown in the figure below. Label each piece as shown.



(b) Fit all the pieces together to make a square whose side is c.



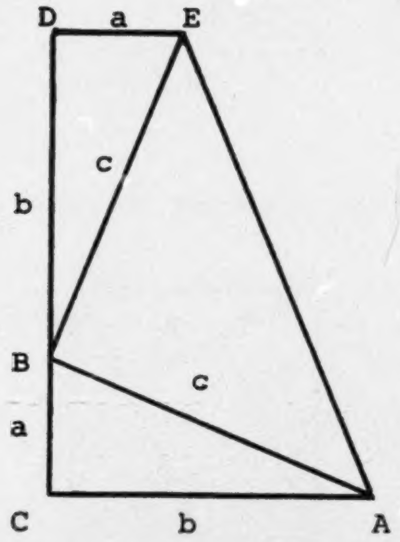
(c) Fit pieces 1, 2, 4, 6, and 7 together to form a square whose side is a.



(d) Fit pieces 3 and 5 together to form a square whose side is b.

Explain how these steps give evidence of the truth of the Pythagorean Theorem.

12. A proof of the Pythagorean Theorem making use of the figure below was discovered by General James A. Garfield several years before he became president of the United States. It appeared about 1876 in the New England Journal of Education.



Prove that $a^2 + b^2 = c^2$ by stating algebraically that the area of the above trapezoid equals the sum of the area of the three triangles ABC, BDE, and EBA. The area of the above trapezoid is found by taking one-half the product of the altitude, which is DC, and the sum of the bases, which are DE and AC. (How do you know EBA is a right angle?)

Answers for Lesson 7.

All answers are given with one decimal place accuracy.

1. (a) c = 5 (b) c = 29.4 (c) a = 15 (d) a = 12.6 (e) b = 9
2. 7.1 inches
3. 9.4 miles
4. 61.8 feet
5. 5.2 inches
6. 7.2 feet
7. 11.2 feet by 22.4 feet
8. 3.0 feet
9. 3
10. 1.4 feet

LESSON 8

Geometry

Besides proving the Pythagorean Theorem on right triangles the Pythagoreans are also credited with the discovery of irrational numbers (irrational numbers are numbers that cannot be expressed as the quotient a/b of two integers a and b, with $b \neq 0$). Until this time it was thought the rational numbers, which contained all fractions and integers, were sufficient for practical measuring purposes. In particular, the Pythagoreans showed that there does not exist a rational number which represents the length of the diagonal of a square of unit one on a side. (Figure 8.1)

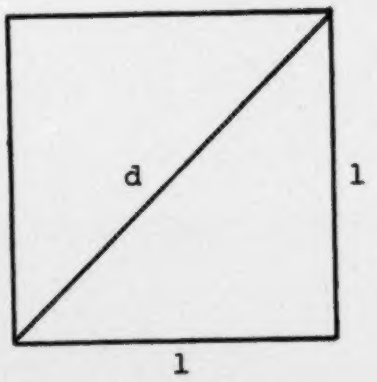


Figure 8.1

From the Pythagorean Theorem we know that

$$d^2 = 1^2 + 1^2 = 2,$$

and since d cannot be expressed as the quotient of two integers then we give d a new symbol and write

$$d = \sqrt{2} .$$

The discovery of the irrationality of $\sqrt{2}$ caused much concern in the

Pythagorean School. Not only did it upset the basic assumption that everything depended on the whole numbers, but it also severely limited one of their basic definitions about proportions which states that given any two numbers x and y there exists positive integers m and n such that

$$mx = ny.$$

This definition was then found to be true only for rational numbers instead of all numbers as they had previously thought. In fact, the discovery of irrational lengths dealt such a blow to the mathematics of the Pythagorean School that a great effort was made to keep the results secret. One source has it that the Pythagorean, Hippasus, was drowned at sea for disclosing this secret to outsiders. Now, this discovery is looked on as one of the great milestones in the history of mathematics.

Using the idea of the diagonal of a square (figure 8.1), the Pythagorean Theorem provides us with a good way of finding certain irrational lengths once we know the unit length. Hence, suppose the following line represents one unit in length, and we wish to construct a line of length $\sqrt{3}$ units.

1 unit

We first construct an isosceles right triangle with 1 for the measure of each leg and the hypotenuse will then be $\sqrt{2}$. Next we construct another right triangle as shown in (figure 8.2).

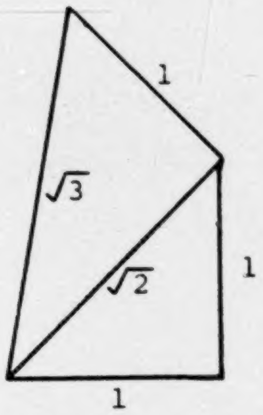


Figure 8.2

Thus, the second triangle has legs of length 1 and $\sqrt{2}$ which means the hypotenuse is of length $\sqrt{3}$ and we have constructed a line of length $\sqrt{3}$. Continuing in this same fashion, right triangles, the measure of whose hypotenuses are

$$\sqrt{4}\ , \sqrt{5}\ , \sqrt{6}\ , \sqrt{7}\ , \cdots,$$

can be constructed. This is called the square root spiral (figure 8.3).

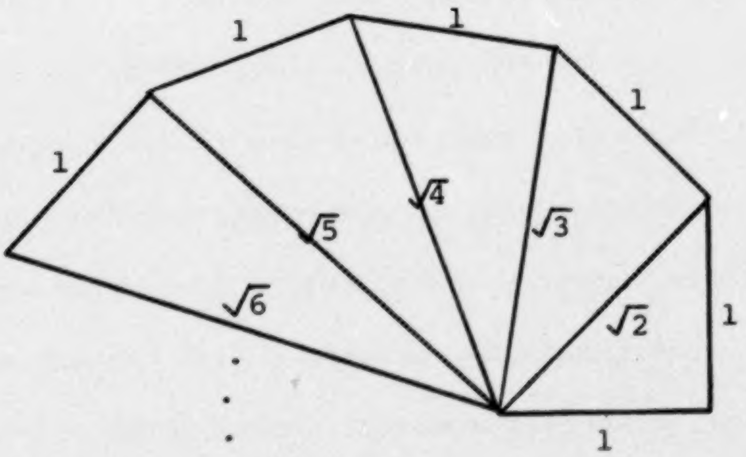


Figure 8.3

Of course, it is impractical to find the square root of fairly large numbers by constructing the spiral, but it can be used in the following sense. For example, suppose we wished to construct a line of length $\sqrt{11}$. This can be done by using the lengths $\sqrt{5}$ and $\sqrt{6}$ found in the spiral of figure 8.3. We have

$$(\sqrt{11}\ )^2 = (\sqrt{5}\ )^2 + (\sqrt{6}\ )^2$$

or

$$11 = 5 + 6.$$

Thus, the construction is shown in (figure 8.4).

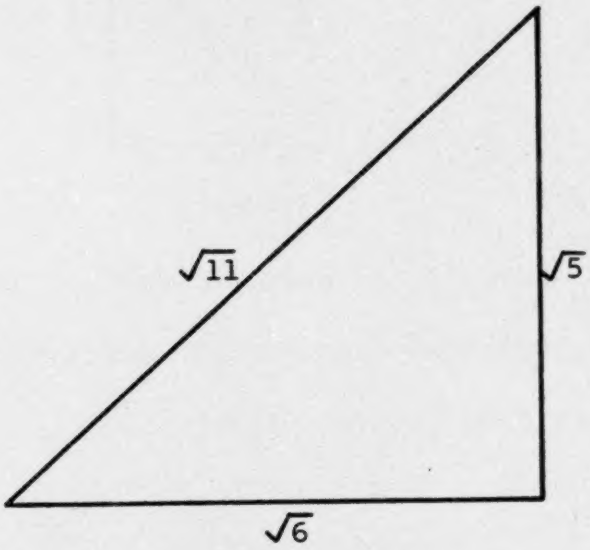


Figure 8.4

Even though the discovery of irrational magnitudes can now be viewed as the most important contribution of the Pythagoreans to mathematics, they devoted most of their effort to studying properties of the whole numbers. They found many interesting results and we shall illustrate by showing some of the properties they proved about figurate numbers. These numbers, considered as the number of dots in certain geometrical configurations, provide a link between arithmetic and geometry. We now define triangular numbers, square numbers, and pentagonal numbers. (figures 8.5, 8.6, 8.7)

Triangular Numbers

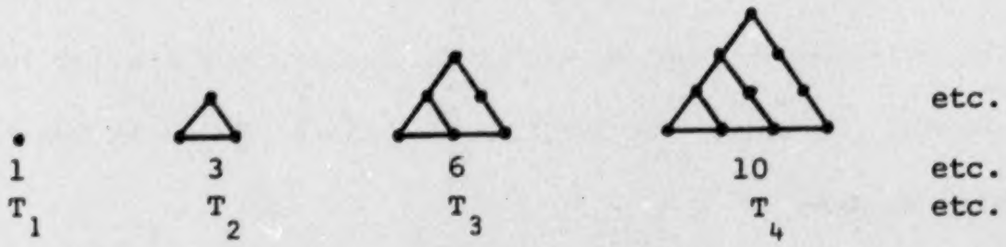


Figure 8.5

Square Numbers

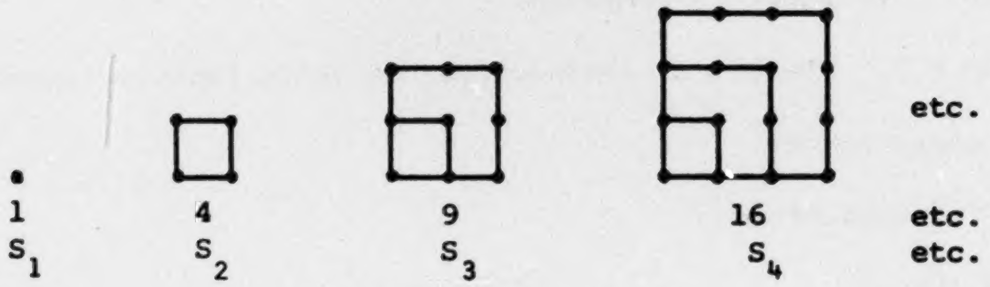

etc.
1
4
9
16
etc.
S_1
S_2
S_3
S_4
etc.

Figure 8.6

Pentagonal Numbers

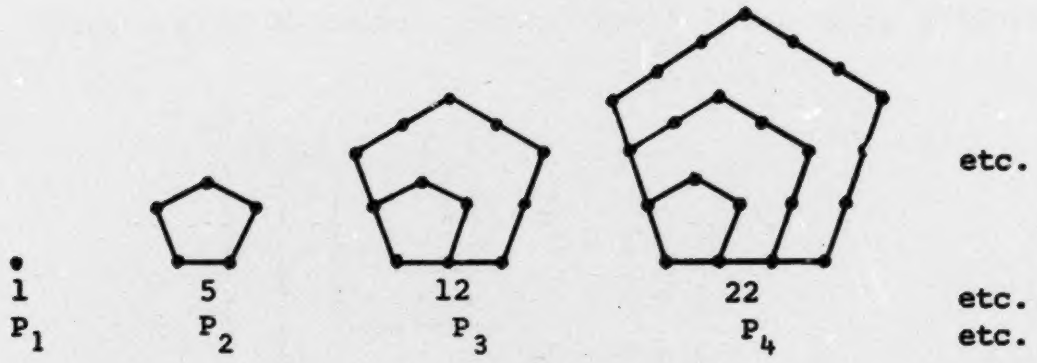
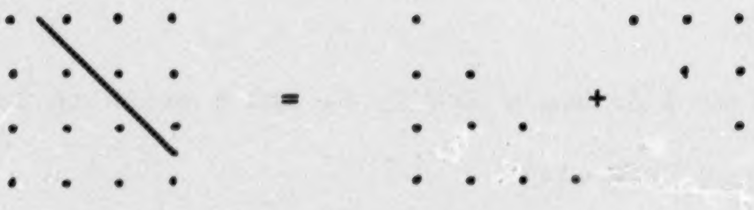

etc.
1
5
12
22
etc.
P_1
P_2
P_3
P_4
etc.

Figure 8.7

Many results about these numbers can be established in a purely geometric fashion.

THEOREM 8.1. Any square number greater than 1 is the sum of two successive triangular numbers.

Geometric Proof:



That is, $S_4 = T_4 + T_3$

Thus, for any square number, $S_n = T_n + T_{n-1}$

by splitting the square number above the main diagonal into two triangular numbers. The proof is complete.

THEOREM 8.2. The sum of consecutive odd whole numbers, starting with 1, is a square number.

Geometric Proof:

Notice that,

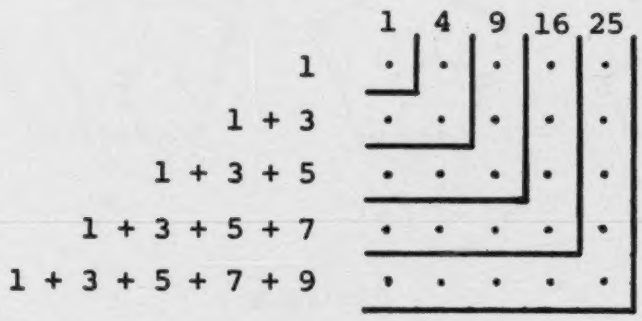
$$1 = 1 = S_1$$

$$1 + 3 = 4 = S_2$$

$$1 + 3 + 5 = 9 = S_3$$

etc.

The following geometrical figure gives a proof of this result.



Exercises for Lesson 8.

1. Construct the square root spiral up to $\sqrt{10}$ using 1 inch as 1 unit.

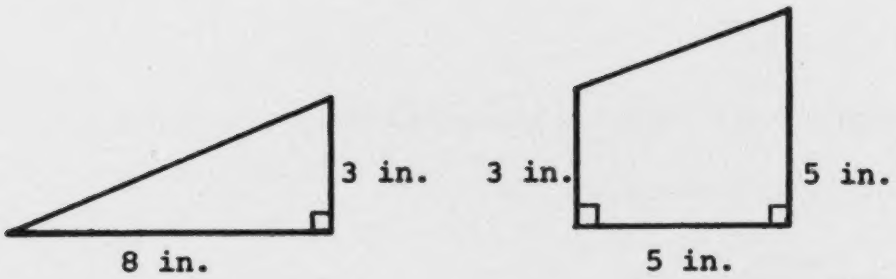
2. Using the square root spiral of problem 1 construct lengths of:

(a) $\sqrt{12}$ (b) $\sqrt{19}$ (c) $\sqrt{16}$ (d) $\sqrt{13}$

3. If the legs of a right triangle are 1 and $\sqrt{m}$ find the hypotenuse of the triangle.

4. The sides of a triangle are 5, 6, and 8 units in length. Is the triangle a right triangle?

5. Cut out four pieces of paper, two each with dimensions given below.



(a) Fit the pieces together to form a rectangle of dimensions 5 in. by 13 in. Find the area of the rectangle.

(b) Find the area of both triangles and trapezoids and add up the results.

(c) Are the results of (a) and (b) the same? Why?

6. Prove the following result for pentagonal numbers. Give a geometric proof. The nth pentagonal number P_n is equal to n plus 3 times the $(n - 1)^{th}$ triangular number. That is,

$$P_n = n + 3\, T_{n-1}.$$

7. An oblong number is the number of dots in a rectangular array having one more column than rows. For example,

• •	• • • • • •	etc.
2	6	etc.
O_1	O_2	etc.

Find the next three oblong numbers.

8. Give a geometric proof that shows that any oblong number is the sum of two equal triangular numbers. (see problem 7)

9. Find

(a) T_5 and T_6.

(b) S_5 and S_6.

(c) P_5 and P_6.

10. Try to discover on your own a theorem about figurate numbers and prove your theorem.

11. Give a geometric proof which shows that 8 times any triangular number, plus 1, is a square number.

Answers for Lesson 8.

3. $\sqrt{1 + m}$
4. no
5. (a) 65 in^2
    (b) 64 in^2
7. 12, 20, 30
9. (a) 15, 21
    (b) 25, 36
    (c) 36, 51

SECTION IV

Statistics

LESSON 9

Statistics

H. G. Wells has predicted that, "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write." That day may well be upon us for we are bombarded with statistics as soon as we can read or write and certainly much sooner than most of us can understand the real significance of the statistical facts we are given. Averages, relationships, graphs, and trends are not always what they seem to be. Like an iceberg, there may be much more there than what we see on the surface or like a fish story there may be a great deal less than what we hear.

Statistics are certainly necessary in our society. Without them and statistical methods, we would be overwhelmed with a mass of data and know very little about what the data tells us. Statistics help us analyze and interpret data. We are all interested in the average value of a set of data. Statistics tell us how to find several averages and which ones best represent the data and which ones do not represent the data at hand. Statistics also tell us something about the variability of data. Is the data widely scattered over many values or does it tend to cluster about a single point? Given two sets of data, which supposedly represent the same thing or event, statistics help us determine just how much the two sets of data are related to each other. Statistics help us in reporting social and economic trends, business conditions, and many many aspects which deal with our daily lives. With all the helpful and useful information with which statistics provide, we might wonder why we hear such phrases as; "There are three kinds of lies: lies, bad lies, and statistics." The main problem seems to lie not so much in the calculation of the statistics themselves, but rather in the way we select our sample to find our statistics, and in the interpretation (misinterpretation many times) of the results we obtain.

An almost classic example of what can happen with a biased sample occurred in 1936 when the Literary Digest predicted a landslide victory for Republican Alfred Landon over Democrat Franklin D. Roosevelt in that year's presidential election. What happened was very simple. The Digest sent out millions of postcards to telephone subscribers all over the nation asking them how they intended to vote. The returns showed a large Republican triumph and based on this information the editors of the Digest felt sure nothing could be wrong since the total response was so large and one-sided. But something was wrong. That something was the fact that during the depression years only the middle and upper classes of people had telephones. So the Digest got its returns from that group in the population most likely to vote Republican, and almost completely ignored the much larger number of people in the poor and lower income class who were going to vote Democratic. Moreover, there is a much greater tendency for people of wealth and education to return mail questionnaires so that the bias in favor of people likely to vote Republican was further enhanced.

As an example of how interpretation of the data can lead to a complete misunderstanding of the situation, let us suppose you are a college graduate and have had several years experience in a small business.

You are given the opportunity to become an executive in the Bully Blanket Factory. You, of course, have talked with many officials of the company and discussed salary arrangements with them. They point out to you with pride that the average salary of workers in this company is $11,667 per year, and your salary will be above average at, say, $12,000 per year with opportunity for advancement in a few years at a substantial increase in salary. At any rate you take the job and in the course of a few months the workers of Bully Blankets decide to unionize. As an executive you do not join the union, and in general are not too much aware of its presence until time for new wage contracts to be made. The president appoints a committee of company executives including you to meet with union officials and consider wage demands. At the first meeting, union leaders point out that the average wage of workers at Bully Blankets is only $7,500 per year, and salaries need to be raised. You, of course, are shocked to find out that salaries are so low, especially since you still remember the nice average $11,667 per year quoted to you a few months earlier. Thus, you feel like someone is lying; either the company president who hired you or else the union leader. But the fact is, neither person lied. They were simply using the average which was favorable to their side. Both figures are legitimate averages, legally arrived at, which represent the same data, the same people, and the same salaries. Yet it is also clear to you that one of them is so misleading as to rival an out-and-out lie.

The trick in each case was to give an unqualified average. The word "average" has at best a very loose definition. The president gave you the mean average since it is the highest in this situation to impress you to come to work for the company. The mean average is found by adding up all the incomes and dividing by the number of workers. The union leader gave you the mode, which is an average found by finding the income that occurs most frequently. Another average which could have been used is the median. The median is the middle value of all the incomes. Thus, if you know the median income then you know that half the workers are making more than this and half are making less. When you are told that something is an average, you don't know very much about it unless you can find which common kind of average it is: mean, median, or mode.

Let us now define the three averages in common use and then see how the results quoted above in the Bully Blanket problem are found.

Averages are called measures of central tendency. The common measures are the arithmetic mean, the median, and the mode. Each of these has certain advantages and limitations. The nature of the data and its intended purpose usually determines which measure of central tendency is used. Sometimes one of the averages represents the data much better than the others while at other times the averages fall so close together that it makes little difference to distinguish between them. This fact sometimes adds to the confusion about the "average."

DEFINITION 9.1. Mean. The arithmetic mean of a set of numbers is found by adding the numbers together and then dividing by the total number of numbers in the set. If there are n numbers in the set denoted as $x_1, x_2, \cdot \cdot \cdot, x_n$ then the mean is denoted by $\overline{x}$ and is given by the following formula:

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdot \cdot \cdot + x_n}{n} .$$

For example, suppose we wish to find the mean of five math tests with scores as given:

$$x_1 = 77$$
$$x_2 = 65$$
$$x_3 = 92$$
$$x_4 = 71$$
$$x_5 = 87$$

Then

$$\overline{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = \frac{392}{5} = 78.4.$$

DEFINITION 9.2. Median. The median of a set of numbers is the middle number when they are arranged in sequential order. If the number of numbers is even then the median is found by finding the mean of the two middle numbers.

The median of

5, 7, 11, 12, 13, 18, 19

is 12 since it is the middle number of the seven numbers given.

The median of

5, 7, 11, 12, 13, 18, 19, 20

is the mean of 12 and 13 or 12.5 since there are two middle numbers.

DEFINITION 9.3. Mode. The mode of a set of numbers is that number that occurs most frequently. If no number occurs more than once then the set of numbers has no mode. If two or more numbers occur the maximum number of times, then each number is called a mode.

For example, the mode of

25, 26, 26, 26, 27, 27, 29, 30, 31, 31, 32

is 26 since it occurs three times in the set of numbers and no other number occurs more than twice.

The modes of

31, 31, 32, 33, 33, 34, 35, 36, 37, 37, 38

are 31, 33, and 37 since each occurs twice in the set and no other number occurs more than once.

PROBLEM 9.4. Find the mean, median, and mode of the following wages paid by the Bully Blanket Company:

Annual Wage	Number Receiving This Wage
$100,000	1
50,000	2
30,000	2
12,000	7
9,000	20
7,500	24
5,000	5
3,000	2

To find the mean we first multiply the annual wage by the number receiving that wage and then total the result.

$$\begin{array}{rcrcr}
1 & \times & 100{,}000 & = & 100{,}000 \\
2 & \times & 50{,}000 & = & 100{,}000 \\
2 & \times & 30{,}000 & = & 60{,}000 \\
7 & \times & 12{,}000 & = & 84{,}000 \\
20 & \times & 9{,}000 & = & 180{,}000 \\
24 & \times & 7{,}500 & = & 180{,}000 \\
5 & \times & 5{,}000 & = & 25{,}000 \\
2 & \times & 3{,}000 & = & \underline{6{,}000} \\
 & & & & 735{,}000
\end{array}$$

There are 63 employees of Bully Blankets so the mean would be

$$\frac{\$735{,}000}{63} = \$11{,}666.67.$$

The median of the 63 values would be the 32nd value which is $9,000. The mode is $7,500. Thus, $11,666.67, $9,000, and $7,500 are all averages of the same data.

Exercises for Lesson 1.

1. Find the mean, median, and mode of each of the following sets.

(a) 55
56
59
61
62
62
65
66
66
70

(b) 2342
2231
2500
1986
2058

(c) 0.076
0.105
0.371
0.401
0.402
0.402
0.403
0.412
0.412

2. Find the mean of the numbers 80 to 100.

3. A contest was held at the Bully Blanket Company to see who could make the most blankets in a given day. Of 17 people in the contest the number of blankets produced was 89, 92, 101, 97, 98, 98, 103, 95, 97, 100, 99, 95, 102, 92, 85, 101, 92.

(a) What was the mean number of blankets produced?

(b) What was the median number of blankets produced?

(c) What was the mode number of blankets produced?

4. On a certain test, the 12 girls in a class made a mean grade of 75 and the 15 boys in the class made a mean grade of 72.

(a) Find the mean of the entire class.

(b) Can you find the median from this information? Why?

5. J. F. Kenny found after grouping, the weights of 1000 eight-year-old girls yielded the following frequency table.

W(lb)	29.5	33.5	37.5	41.5	45.5	49.5	53.5	57.5	61.5	65.5
F(frequency)	1	14	56	172	245	263	156	67	23	3

(a) Find the mean weight of eight-year-old girls?

(b) Find the median weight of eight-year-old girls?

(c) Find the mode weight of eight-year-old girls?

6. Try to make up a set of grades of 10 students, 9 of whom have grades below the mean. In this situation, is almost everyone below average?

7. In baseball one speaks of a player's "batting average." Is this one of the averages we have discussed above? If so, which one?

8. Count the number of words in each line of page 3 of this book. Arrange your data to show the number of lines having 1, 2, 3, · · · words. Find the mean, mode, and median for your data. Check with a friend and see if he got the same results as you did. If not, why are your results different?

9. A student has received grades of 96, 87, and 47 on three tests. What grade must he get on the next test in order to have a mean grade of 80?

10. An instructor increased the scores on each test paper of a certain test by 7 points. What did this do to the class average or mean?

11. Suppose the arithmetic mean of the grades of the members of a certain class is computed. What other average would you need to know to determine whether at least half the class has grades above the mean?

12. In terms of averages, what does the following statement mean? "He is in the upper 1/3 of his class."

Answers for Lesson 9.

1. (a) 62.2 mean, 62 median, 62 and 66 mode
   (b) 2223.4 mean, 2231 median, no mode
   (c) 0.3316 mean, 0.402 median, 0.402 and 0.412 mode
2. 90

3. (a) 96.2 mean, (b) 97 median, (c) 92 mode
4. (a) 73.3 (b) No, the individual scores are not known.
5. (a) 47.7 lbs., (b) 49.5 lbs., (c) 49.5 lbs.
7. Yes, mean average
9. 90
10. Increased mean by 7 points
11. Median

LESSON 10

Statistics

In general, the need for averages is a sign of a very definite and legitimate need to characterize a whole collection of data by a single number. Imagine trying to tell someone about incomes of citizens of the United States without resorting to some kind of an average. The sheer bulk of the data would make it impossible for the mind to comprehend. Although an average can be misleading as we have seen in the previous lesson, many times it can give a very accurate idea of a trend in a set of data and describe the situation in a very short and well-defined manner. There are, of course, situations where one average "better" or "best" represents the collection of data than others. This means that in order to select the average or averages which "best" represent a collection of data we need to know something about the advantages and limitations of the various averages.

The arithmetic mean is the most commonly used average or measure of central tendency. We will now look at some of the advantages of the mean over the other averages. It takes all numbers into consideration; it is a well-defined average for algebraic manipulation (that is, it is the only one expressible as a formula); and it is usually the best possible single number to use if the data is fairly uniform. The mean is the value each measure would have if all the values of the collection of data were equal. One rather big limitation of the mean is the fact that it is strongly influenced by extreme values as we saw in the Bully Blanket problem. A situation where the mean would probably represent the best average can be found in computing the average height of third graders.

The median measure of central tendency is a positional average. It tells us that there are just as many values above the median as there are values below. It is the middle value. The advantages of the median are found in the fact that it is very easy to compute and it is not influenced by extreme values of the data. For example the median of

5, 7, 7, 8, 9, 10, 100

is 8 which certainly better represents this data than the mean which is 20.9. A limitation of the median is found when the data is concentrated in distinct and widely separated groups. For example, the median of

5, 7, 7, 8, 61, 63, 65

is 8 which doesn't give a very good average of this set of data. An example where the median would probably represent the best average can be found in computing the average income of persons living in Brazil.

The mode is probably the least used of the measures of central tendency. It is thought to be the most typical measure of all, since it occurs with the most frequency. It is most useful when the collection of data is very large. The advantages of the mode can be found in the fact that it is very easy to compute and it is representative of the data as the most usual value. For example, a shirt manufacturer may find it more useful to know that more men wear a 15½ collar size than any other collar size than to know that the mean collar size is 15.28. A disadvantage of the mode is the fact that it does not take account of the other values in the data.

Thus, when we are given an average we should ask what kind of average it is, and try to determine within our own mind whether or not this average "best" represents the situation being described to us. Remember also that there are some collections of data where no one of the averages "best" describes the set. For example, consider the following set:

5, 6, 6, 7, 7, 25, 31, 31, 31.

The mode is 31, median is 7, and the mean is 17. Which of these numbers best represents the set?

Exercises for Lesson 10.

1. A man working for a company complained that although the company asserted that the "average" pay of its employees was $90 per week, he got only $72 per week and was the highest paid man in his department. What are the possibilities for accounting for this apparent discrepancy?

2. An experimental plot of hybrid corn produced the following heights in inches:

48 57 64 73 72 53 62 61

74 78 72 71 81 71 73 71

Find the mean, median, and mode. Which of these averages do you consider "best" represents the experimental plot?

3. Suppose you are a buyer for a shoe store. Which type of average will probably be of greatest benefit to you?

4. The following wages are paid by the B & B Company:

Annual Wage	Number Receiving this Wage
15,000	2
6,000	3
4,500	4
4,000	6

(a) Find the mean, median, and mode wage.

(b) What average would the employees use in trying to get a raise?

(c) What average would the B & B Company use in advertising for help?

(d) What average would the Internal Revenue Bureau use in speaking of taxable income?

(e) What average do you feel best represents the typical wage?

5. What average would be used in the following:

(a) In meteorology, for finding the average rainfall in the state of Oklahoma.

(b) In chemistry, for finding the molecular weight of a metal.

(c) In a department store, for determining the number of shirts of each size to stock in the store.

(d) In business, for computing average wages.

(e) In psychology, for determining the average IQ.

6. The contributions by friends of a college to provide a parking lot are as follows:

Amount in Dollars	Number Giving This Amount
10,000	1
1,000	2
500	3

(cont'd)

Amount in Dollars	Number Giving This Amount
100	10
50	15
25	18
15	5
10	7

(a) Find the mean, median, and mode.

(b) What is the "usual" contribution?

(c) What average would best represent this data?

Answers for Lesson 10.

2. Mean - 67.6
   Median - 71
   Mode - 71
3. Mode
4. (a) Mean - $6,000
       Median - $4,500
       Mode - $4,000
6. (a) Mean - $259.75
       Median - $50
       Mode - $25

LESSON 11

Statistics

The big day has finally arrived: the start of basketball season and the first big game with Silo Tech. You, as coach of the Teasippers, are not too worried since the mean height of the twelve members on your team is 6 ft. 3 in. (75 in.) and you have fairly good ball control and shooting ability. You have been told that the mean height of the eleven members of the Tech team is 6 ft. 2 in. (74 in.) and feel pretty good about that. However, what you don't know is that the Silo Tech coach has been keeping Richard Huge, 7 ft. 4 in. (88 in.) their most promising player, a secret. Of course, as soon as you see Richard Huge you wonder how the mean height of the team can be so small when Richard is so tall, and you check the heights of the players again.

Here is a list of the heights, in inches, of the eleven members of the Silo Tech team, in order from shortest to tallest:

64, 66, 70, 72, 73, 74, 75, 76, 78, 78, 88.

Sure enough the mean is 74 inches.

The list of heights of your team in inches, again from shortest to tallest is:

71, 72, 73, 73, 74, 75, 75, 76, 77, 77, 78, 79.

The mean is 75 inches.

Thus, it becomes quite clear to you that while the means are very close to the same number there is something quite different in the two

lists of heights. That something is variability. The Silo Tech team has much more variability in height than does your team. Their shortest man, when averaged in with Richard, tends to offset the extreme heights of either player. What is needed is to have some sort of statistical measure which tells us of the variability of a set of data.

There is in common use two measures of variability. The first of these is called the range and is very easy to compute.

DEFINITION 11.1: Range. The range of a set of numbers is the difference between the largest and smallest number in the set.

Thus, the Silo Tech team varied from a short 64 in. to a tall 88 in. which gives a range of 24 in. While the Teasippers varied from 71 in. to 79 in. which gives a range of 8 in. If you had known the range of the heights of each team then you would have been expecting some rather tall players from Silo Tech.

The range has the advantage of being the easiest of the two common measures of variability to compute, but as can be seen from the following example it is at times misleading since it depends only on two numbers in the set. Consider the two sets of numbers S_1 and S_2.

S_1: 25, 26, 26, 35, 42, 44, 45

S_2: 25, 34, 35, 35, 36, 36, 45.

The means of S_1 and S_2 respectively are 34.7 and 35 whereas the range of both sets is 20. Hence, even though there appears to be more variability in S_1 when compared to S_2 the range does not indicate this to us. The second measure of variability is called the standard deviation and is usually denoted by the symbol sigma, σ.

DEFINITION 11.2. Standard Deviation. The standard deviation of a set of n numbers denoted as $x_1, x_2, \cdot \cdot \cdot, x_n$ is found by the following formula:

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdot \cdot \cdot + (x_n - \bar{x})^2}{n}} .$$

The standard deviation is a very reliable measure of variability. It takes all numbers into account. When it has a small value this usually means the numbers are not widely scattered and tend to cluster around the mean. When the standard deviation is large this means that the numbers have a great deal of variability and do not tend to cluster around the mean. Thus, a large standard deviation is usually associated with much variability of the data whereas a small standard deviation usually means small variability of the data.

In finding the standard deviation of a set of data we must first find the mean of the data, then subtract it from each value and square this difference. We then sum these squares and divide the sum by the number of values. The final result is then obtained by taking the square root of the quotient.

Let us find the standard deviation of the sets S_1 and S_2. Table 11.1 gives the method for finding the standard deviation σ_1 of set S_1. Table 11.2 gives the method for finding the standard deviation σ_2 of set S_2.

Value	Deviation from Mean (34.7)	Deviation Squared
25	-9.7(25 - 34.7)	94.1
26	-8.7(26 - 34.7)	75.7
26	-8.7	75.7
35	.3	.1
42	7.3	53.3
44	9.3	86.5
45	10.3	106.1
n = 7		491.5

$$\sigma_1 = \sqrt{\frac{491.5}{7}} = \sqrt{70.2} \approx 8.4.$$

Table 11.1

Value	Deviation from Mean (35)	Deviation Squared
25	-10(25 - 35)	100
34	-1(34 - 35)	1
35	0	0
35	0	0
36	1	1
36	1	1
45	10	100
n = 7		203

$$\sigma_2 = \sqrt{\frac{203}{7}} = \sqrt{29} \approx 5.4.$$

Table 11.2

Comparing the two numbers, $\sigma_1 = 8.4$ and $\sigma_2 = 5.4$ we see that set S_1 with the higher standard deviation has more variability in it than does set S_2. That is, the values of S_1 have a greater tendency to diverge from the mean score.

Let us now compute the standard deviations for the heights of the basketball players from Silo Tech and the Teasippers. We first compute σ_s for Silo Tech found in Table 11.3.

Height	Deviation from Mean (74)	Deviation Squared
64	-10	100
66	-8	64
70	-4	16
72	-2	4
73	-1	1
74	0	0
75	1	1
76	2	4
78	4	16
78	4	16
88	14	196
n = 11		418

$$\sigma_s = \sqrt{\frac{418}{11}} = \sqrt{38} \approx 6.2.$$

Table 11.3

The standard deviation for the Teasippers is denoted by σ_t and is found in Table 11.4.

Height	Deviation from Mean (75)	Deviation Squared
71	-4	16
72	-3	9
73	-2	4
73	-2	4
74	-1	1
75	0	0
75	0	0
76	1	1
77	2	4
77	2	4
78	3	9
79	4	16
n = 12		68

$$\sigma_t = \sqrt{\frac{68}{12}} = \sqrt{5.6} \approx 2.4$$

Table 11.4

From these results we see that the standard deviation for Silo Tech is much higher than the standard deviation of the Teasippers. Thus, we would expect more variability of heights among the Silo Tech team. If the coach had known the range and standard deviation of heights of players from Silo Tech as well as the mean he would have been expecting one or two tall players on their team.

Computing the standard deviation seems to be a difficult job, but many times it is the only measure which tells us the true variability of a set of data. It is also quite useful in interpreting results from the data. For example, suppose the mean score on one mathematics test is 70 with a standard deviation of 8. While the mean score on another mathematics test is 75 with a standard deviation of 6. Suppose you made 78 on both tests. Did you do equally well on both tests or better on one of them? On the first test with a score of 78 you scored 8 points above the mean or 1 standard deviation above the mean. On the second test with a score of 78 you scored 3 points above the mean or only 1/2 standard deviation above the mean since on this test the standard deviation was 6. Therefore, you scored better on the first test.

Another reason the standard deviation is an important number to know is that in any set of data which forms a normal distribution approximately 68% of the data is within 1 standard deviation of the mean, while about 95% of the data is within 2 standard deviations of the mean. A normal distribution of data occurs in many kinds of measurements. By a normal distribution we mean that about the same number of values of the data will occur below the mean as occurs above the mean and that the values are not too widely scattered. Consider the following problem.

PROBLEM 11.3. A chemist has an element which he suspects is cobalt. The density of cobalt is 8.9. By careful analysis he finds the measurements of densities of the element to be a normal distribution of data with a mean density of 7.5 and standard deviation of .5 . If his measurements are correct what should he conclude about the element? Why?

Solution: He should conclude that the element is not cobalt since the distribution is normal and in any normal distribution approximately 95% of all values of the data will be within 2 standard deviations of the mean. Thus at least 95% of the values he obtained are not the same as the density of cobalt.

Exercises for Lesson 11.

1. Compute the range and standard deviation for the following numbers.

12, 14, 17, 15, 18, 16, 20, 16

2. In a class of 12 students the scores on a test are as follows:

76, 82, 66, 77, 75, 93, 97, 81, 95, 67, 83, 80

Find the mean, median, mode, range, and standard deviation.

3. Find the standard deviation of weights of 1000 eight-year-old girls.

W(lb)	29.5	33.5	37.5	41.5	45.5	49.5	53.5	57.5	61.5	65.5
F(frequency)	1	14	56	172	245	263	156	67	23	3

Is this set of data a normal distribution?

4. An electrical company is interested in a delayed timing switch which will open in 3 minutes plus or minus 1 second. That is, 180 ± 1 seconds. Two companies, A and B, each submit 10 samples of a delayed timing switch for testing. The results of the time the various switches opened in seconds are as following:

Company A	Company B
179.8	180.6
180.5	179.8
179.6	179.6
179.4	179.2
180.0	179.5
179.9	180.5
180.3	180.2
180.2	179.1
180.5	180.9
179.8	180.6

(a) Find the mean time for both companies.

(b) Find the standard deviation for both companies.

(c) If the bids of both companies were about the same which company would you select to make the switches?

(d) If the bid of Company B was quite a bit lower than Company A might the answer in part c be changed? Explain.

5. Two tests are given. On the first the mean was 75 and the standard deviation was 8. On the second the mean was 79 and the standard deviation was 10.

(a) If you made 79 on each test on which one did you do better?

(b) Suppose you made 83 on the first test and 89 on the second test. On which test did you do better?

(c) Suppose you made 73 on the first test and 74 on the second test. On which test did you do better?

(d) Suppose you made 71 on the first test and 76 on the second test. On which test did you do better?

6. If you are three inches taller than the average person your age and the standard deviation is 2, how many standard deviations are you taller than the average person?

7. Two history tests are given. On the first test the mean was 54 and the standard deviation 8. On the second test the mean was 28 and the standard deviation 10. The scores of students on the tests are as follows:

Student	First Test	Second Test
J	62	38
K	66	23
L	46	28
M	54	33
N	70	43

(a) Which students improved on the second test?

(b) Which students came down on the second test?

(c) Which students remained the same on both tests?

(d) Which student dropped the most on the second test?

8. Suppose you bought a ring from a friend who told you it was platinum, but you were suspicious. You asked a chemist to run a density check on the ring and after ten tries he finds the mean density is 17.25 with a standard deviation of .25. The density of white gold is 17.05 and the density of platinum is 21.45. Are your suspicions well founded? If

the mean density had been 19 with a standard deviation of 1.3 would your suspicions have been well founded?

9. New car A has a mean life of 7 years and standard deviation of 3 years, whereas new car B has a mean life of 8 years and standard deviation of 1 year. Which car would you buy, assuming all other factors are equal? Explain.

Answers for Lesson 11.

1. Range - 8, $\sigma = 2.3$
2, $\overline{x} = 81$, median = 80.5, no mode, range = 31, $\sigma = 9.6$
3. $\sigma = 5.9$, yes
4. For company A, $\overline{x} = 180$, $\sigma = .35$.
   For company B, $\overline{x} = 180$, $\sigma = .61$.
5. (a) 1st test
   (b) neither
   (c) 2nd test
6. 1½
7. (a) L and M
   (b) K and N
   (c) J
   (d) K

SECTION V

Probability

LESSON 12

Probability

A couple named Johnson have two daughters. They have always considered a family of two boys and two girls as ideal. Recently Bob remarked, "After the girls were born we began to doubt that we'd come out with two girls and two boys. But I have figured it out now, since our chances were fifty-fifty in the beginning, of course they still are, since boys are just as likely as girls." The Johnson's situation illustrates a basic problem in the probability of even chances, but is Bob's solution correct?

To test his hypothesis let's consider the chances that a couple planning for two children will have one boy and one girl.

Assuming that boys and girls are born with equal frequency, each child is as likely to be a boy as a girl. Now let's name all the possibilities: boy and girl, two boys, two girls, all equally likely? No! What we actually have are four equal possibilities. They are: boy-boy (BB), girl-girl (GG), boy-girl (BG), girl-boy (GB). So we have four equal possibilities which we call OUTCOMES: BB, GG, BG, GB. The set S of possibilities or outcomes we call a SAMPLE SPACE, thus

$$S = \{BB, GG, BG, GB\}.$$

Now for our problem we are interested in the chances of a boy and a girl. That is, we are interested in a SUBSET, E_1 of our sample space S composed of one boy and one girl,

$$E_1 = \{BG, GB\}.$$

Of course there are other subsets. We might consider the subset of two girls E_2:

$$E_2 = \{GG\},$$

or the subset of two boys E_3:

$$E_3 = \{BB\}.$$

Maybe even the subset of at least one boy being born, E_4:

$$E_4 = \{BB, BG, GB\}.$$

Such subsets, when defined by a special condition are called EVENTS in the sample space. For example, the event E_5, "of two children to be born into a family at least one will be a girl" is

$$E_5 = \{BG, GB, GG\},$$

and the set of all outcomes (sample space) is

$$S = \{BB, GG, BG, GB\}.$$

So, we see that two of the four outcomes consist of a boy and a girl. Thus, there is an even "chance" of a mixed pair in two children. However, instead of the term "chance" we use the expression "PROBABILITY". In a situation with several possible outcomes all equally likely we define probability to be the number of outcomes favorable to the event divided by the total number of possible outcomes.

DEFINITION 12.1:

$$\text{Probability of an Event} = \frac{\text{Number of outcomes favorable to the event}}{\text{Total number of possible outcomes}}.$$

From this definition we see that probability will be a number between 0 and 1. If an event is certain, its probability is 1. If an event cannot happen, its probability is 0. Now,

$$\text{Probability of one girl and one boy} = \frac{\text{Number of outcomes in } E_1}{\text{Number of outcomes in } S}$$

or we denote this

$$P(E_1) = \frac{1}{2} .$$

But what about the Johnson's plans for two boys and two girls? First we list all the possible outcomes (our Sample Space S):

$$S = \{BBBB, BBBG, BBGB, BBGG, BGBB, BGBG, BGGB, BGGG, GBBB, GBBG, GBGB, GBGG, GGBB, GGBG, GGGB, GGGG\}.$$

The total number of possible outcomes is 16. Next we list all outcomes favorable to the event E, "two boys and two girls":

$$E = \{BBGG, BGBG, BGGB, GBBG, GBGB, GGBB\}.$$

The total number of outcomes which are favorable to our event E is 6. So,

$$\text{Probability of 2 girls and 2 boys} = \frac{6}{16} = \frac{3}{8} ,$$

or

$$P(E) = \frac{3}{8} .$$

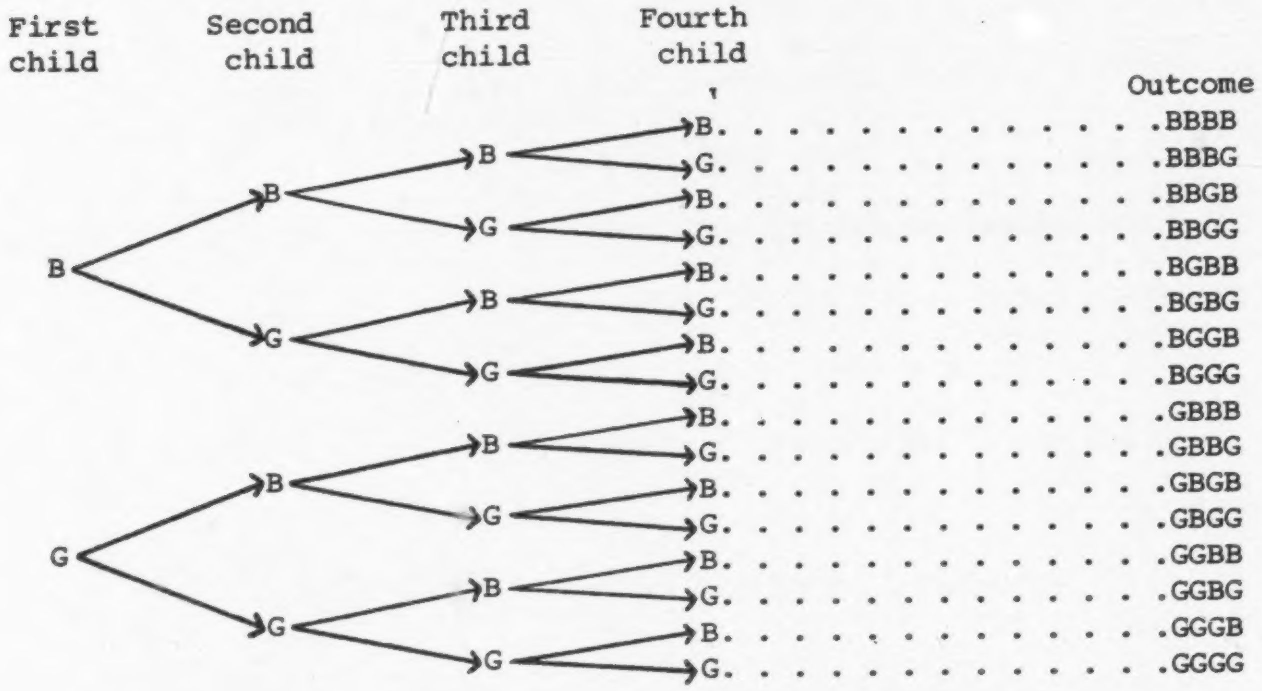
which is less than $\frac{1}{2}$. So we see Bob is wrong, the "chances" are less than fifty-fifty of having 2 boys and 2 girls.

This particular problem can also be analyzed in a slightly different way. Notice that each child to be born into the family of four has two possibilities, boy or girl. This is illustrated by the following table:

Child Number	1	2	3	4
Possibilities:	Boy or Girl	Boy or Girl	Boy or Girl	Boy or Girl

Probabilities which are based on outcomes in which each part of an outcome has only 2 possibilities are called BINOMIAL PROBABILITIES.

One method for finding all possible outcomes of this type is to use a diagram called a TREE. To illustrate, consider again our problem of a family planning for 2 boys and 2 girls.


First child
Second child
Third child
Fourth child
Outcome
B
G
B
G
B
G
B
G
B
G
B
G
B
G
B
G
B
G
B
G
B
G
B
G
B
G
.BBBB
.BBBG
.BBGB
.BBGG
.BGBB
.BGBG
.BGGB
.BGGG
.GBBB
.GBBG
.GBGB
.GBGG
.GGBB
.GGBG
.GGGB
.GGGG

The right hand column of the tree contains the 16 possible outcomes. Such a diagram is very useful for generating the sample space, but there is even a better way. It involves a pattern of numbers called PASCAL'S TRIANGLE, known for centuries and named after a French mathematician of the 17th century.

Pascal's Triangle is as follows:

1 1

1 2 1

1 3 3 1

1 + 3 3 + 3 3 + 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

. . . .

Each number within the triangle is found by adding the pair of numbers directly above it at the left and right. Additional rows of numbers can be included in the triangle by first writing a 1 at each end of the row and then adding the pairs of numbers in the previous row.

Now to illustrate how Pascal's triangle can be used, note the second row:

1 2 1

This row represents the number of ways in which the sex of two children born to a family can turn out.

Number of Boys (or Girls)	2	1	0
Number of Ways	1	2	1
Probability	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

where,

$$\text{Probability of an Event} = \frac{\text{No. of outcomes favorable to the event}}{\text{Total possible outcomes}}$$

and total possible outcomes = 1 + 2 + 1

= 4.

The third row of the triangle

1 3 3 1

represents the numbers of ways in which a family of 3 children may be grouped by sex.

Number of Boys (or Girls)	3	2	1	0
Number of Ways	1	3	3	1
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Finally the fourth row of the triangle

1 4 6 4 1

represents the problem we are considering, the number of ways in which a family of 4 children may be grouped by sex.

Number of Boys (or Girls)	4	3	2	1	0
Number of Ways	1	4	6	4	1
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

To find our probability, we move along the row labeled "Number of Boys" to the number 2. Going down this column we note first the number 6, which is the number of different ways in which 2 of the children born into a family of 4 could be boys. The next number in this column is $\frac{6}{16}$, the probability of exactly 2 boys out of a family of four children.

Now let's consider another problem. Suppose we roll a die which is perfectly balanced, and is as likely to fall with one face showing as another. There are six possible outcomes: 1, 2, 3, 4, 5, 6. The sample space S is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

What is the probability of the event E_1, "rolling an even number with a die"?

$$E_1 = \{2, 4, 6\}.$$

So we have for the probability of the event E_1.

$$P(E_1) = \frac{3}{6} = \frac{1}{2}.$$

What is the probability of rolling a 7 with a die? Let E_2 be the event "rolling a seven with a die", then

$$E_2 = \{\ \}.$$

Such sets as E_2 which contain no elements are called EMPTY sets or NULL sets. Since it contains no element then

$$P(E_2) = \frac{0}{6} = 0.$$

We see the probability of an impossible event is 0.

What is the probability of the event E_3, "rolling a number less than 10 with a die?"

$$E_3 = \{1, 2, 3, 4, 5, 6\}.$$

And

$$P(E_3) = \frac{6}{6} = 1.$$

We see the probability of an event that is certain to happen is 1.

Let's try one more. What is the probability of the event F, "rolling a six" with a die?

$$F = \{6\}$$

then

$$P(F) = \frac{1}{6}.$$

Exercises for Lesson 12.

1. Suppose you toss a coin. List the possible outcomes of the sample space.

2. If you toss a coin, what is the probability of the event:

   (a) heads

   (b) tails

3. If you roll a die, what is the probability of the event:

   (a) a 3 showing?

   (b) an odd number showing?

   (c) a number less than 5 showing?

   (d) a number more than 6 showing?

4. Suppose you draw a card from a deck of cards. What is the probability that you will draw:

   (a) an ace?

   (b) a king of hearts?

(c) a spade?

(d) a face card (jack, queen, or king)

5. A piggy bank contains 30 pennies, 25 nickels, 10 dimes, and 5 quarters. If it is equally likely that anyone of the coins will fall out when the bank is turned upside-down and shaken, what is the probability that the coin:

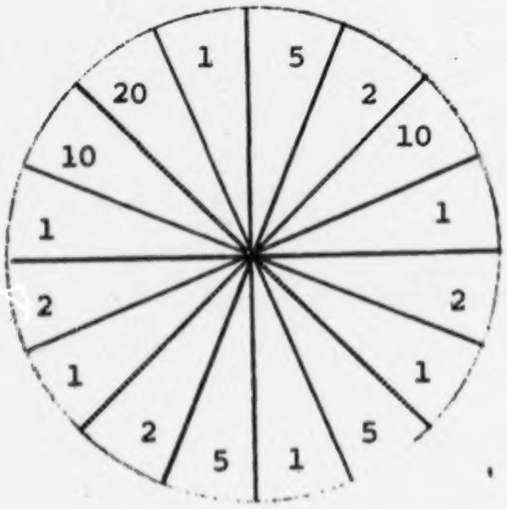
(a) will be a penny?

(b) will be either a nickel or a quarter?

(c) will not be a quarter?

(d) would be at least enough to pay for a phone call?

6. At a gambling casino there is a wheel with the numerals 1, 2, 5, 10, and 20 marked on it. The wheel is shown as follows:



What is the probability of the following events:

(a) 1 will appear under the pointer?

(b) 2 will appear under the pointer?

(c) 5 will appear under the pointer?

(d) 10 will appear under the pointer?

(e) 20 will appear under the pointer?

7. Copy the seven rows of Pascal's Triangle shown in this lesson and then add two more rows.

8. Use Pascal's Triangle to determine:

(a) the number of ways one head may turn up when 2 coins are tossed.

(b) the probability of getting two heads.

9. Use Pascal's Triangle to determine:

(a) the number of ways 5 heads may turn up when 6 coins are tossed.

(b) the probability that 5 heads will turn up when 6 coins are tossed.

10. If ten coins are tossed, what is the probability that ten heads will turn up?

11. In a family with three children, what is the probability that all three will be boys?

12. In a family with three children, what is the probability of there being at least one girl?

13. Suppose the eleven letters of the word "mathematics" are written on cards and the cards are shuffled. If a card is randomly chosen, what is the probability that it will contain:

(a) the letter M?

(b) a vowel?

(c) a letter in the word "misery"?

Answers for Lesson 12.

1. heads, tails
2. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$
3. (a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) 0
4. (a) $\frac{1}{13}$ (b) $\frac{1}{52}$ (c) $\frac{1}{4}$ (d) $\frac{4}{13}$

5. (a) $\frac{3}{7}$ (b) $\frac{3}{7}$ (c) $\frac{13}{14}$ (d) $\frac{3}{14}$

6. (a) $\frac{3}{8}$ (b) $\frac{1}{4}$ (c) $\frac{3}{16}$ (d) $\frac{1}{8}$ (e) $\frac{1}{16}$

7.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

. . . .

8. (a) 2 (b) $\frac{1}{4}$

9. (a) 6 (b) $\frac{3}{32}$

10. $\frac{1}{1024}$

11. $\frac{1}{8}$

12. $\frac{7}{8}$

13. (a) $\frac{2}{11}$ (b) $\frac{4}{11}$ (c) $\frac{5}{11}$

LESSON 13

Probability

Suppose that as you are walking down the street in a "shady" part of town, you hear such expressions as "seven come eleven", and "uh-oh, snake-eyes" coming from a dark alley. Being a curious type and very brave, you walk over to a group of "grungy" looking characters and ask what they are doing. One particularly mean looking fellow with a pair of dice in his hands grunts, "shootin' craps." Before you have much chance to think about it, the mean one thrusts the dice, one white and one red, into your hands and says, "tell ya what we'll do, we'll give you one toss of the dice. If you throw snake-eyes (2-ones) we're gonna hit ya over the head an' take all yer money. If you throw a seven or eleven ya can walk out of this alley free. If ya roll anything else we'll just take your money and let ya go." As you squat there with the dice in your hand, perspiration rolling off your face, wondering what your chances are of getting out of this mess without getting banged on the head and losing your money, you suddenly recall a bit of knowledge picked up in mathematics about how to figure your chances in just such a situation. But, doggone it, how do you form those sample spaces? Since these guys are really enjoying watching a "big time" college grad like you shake all over, they don't seem to mind as you quickly scratch your sample space into the dirt. The best you can come up with is the following diagram.

WHITE DIE

RED DIE	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

Now recalling your chances of walking out untouched are the number of ways you can get seven or eleven, you look over the diagram and begin counting all the "favorable" outcomes. There are six ways to get seven and two ways to get eleven. A total of 8 "favorable" ways to roll the dice out of 36 possible ways. So your chances are: $\frac{8}{36}$.

Another way which you could have analyzed the problem would be to observe that the event E, "seven or eleven" is actually composed of two distinct events, the event E_1, "throwing a seven",

$$E_1 = \{(16), (25), (34), (43), (52), (61)\}$$

and the event E_2, "throwing an eleven",

$$E_2 = \{(56), (65)\}$$

where (16) means a 1 appears on the red die and a 6 appears on the white die.

The event E, "throwing a seven or throwing an eleven" can be written

$$E = E_1 \text{ or } E_2,$$

or more frequently the connective "or" is replaced by the symbol $\cup$, and the event E is written

$$E = E_1 \cup E_2.$$

This is read "E_1 union E_2" or simply "E_1 or E_2". Since the events E_1, "rolling a seven", and E_2, "rolling an eleven" cannot both occur simultaneously, we say that E_1 and E_2 are MUTUALLY EXCLUSIVE EVENTS, and for such events the probability of one "or" the other of these events occurring is

THEOREM 13.1: $P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

That is, if we knew the probability of "rolling a seven", $P(E_1)$, and the probability of "rolling an eleven", $P(E_2)$, then the probability of rolling a "seven or an eleven", $P(E_1 \cup E_2)$, would simply be the sum of these respective probabilities.

In this case

$$P(E_1) = \frac{\text{No. of outcomes producing 7}}{\text{Total no. of possible outcomes}}$$

$$P(E_1) = \frac{6}{36},$$

and

$$P(E_2) = \frac{\text{No. of outcomes producing 11}}{\text{Total no. of possible outcomes}}$$

$$P(E_2) = \frac{2}{36}.$$

Thus,

$$\begin{aligned} P(E) &= P(E_1 \cup E_2) \\ &= P(E_1) + P(E_2) \\ &= \frac{6}{36} + \frac{2}{36} = \frac{8}{36}, \end{aligned}$$

which is what had previously been determined.

In any case your chances are not too good. Well, what about your chances of a severe headache and no money? You get that with "Snake-eyes" (when you have a sum of 2), which is $\frac{1}{36}$, a fairly good chance that you might miss the headache. Finally you wonder what your chances are of getting out with just losing your money--that's when you throw anything besides "seven or eleven" or "snake-eyes". Looking over your sample space you see there are 27 outcomes which favor this event. So the probability of "not throwing seven or eleven or snake-eyes" is $\frac{27}{36}$. Chances are good that you're going to lose your money. As your eyeballs begin to bulge and that funny itchy feeling begins creeping up your neck, you fight to regain your "cool" and think of a way to improve your chances of getting out safely. You suddenly make the observation that getting a 6 or 8 is more likely than 7 or 11 since

$$\text{Probability of 6 or 8} = \frac{10}{36} .$$

Not much advantage, but $\frac{10}{36}$ is better than $\frac{8}{36}$, the probability of "rolling a seven or eleven".

So you try, "Tell you what, 7 and 11 are unlucky numbers for me, would it be alright to try for a 6 or an 8?" Not capable of reading your sample space or comprehending your magnificent shrewdness, they agree.

One word of caution is in order here, you do have to be certain that the events E_1 and E_2 are mutually exclusive events, i.e. cannot occur simultaneously, before applying the formula

$$P(E_1 \cup E_2) = P(E_1) + (E_2).$$

One way of determining whether two events are mutually exclusive is to observe whether the event E_1, and the event E_2 have any elements (outcomes) in common. If they do not have any common elements, they are mutually exclusive events. The set of elements common to E_1 and E_2 is denoted

$$E_1 \cap E_2,$$

which is read: "the intersection of E_1 and E_2" or "the event E_1 and E_2".

For example, let E_1 represent the event "the sum is greater than 9", and E_2 represents the event "the sum is an odd number greater than 8". Then we have

$$E_1 = \{(46), (55), (64), (56), (65), (66)\},$$

$$E_2 = \{(36), (45), (54), (63), (56), (65)\}$$

and

$$E_1 \cap E_2 = \{(56), (65)\}.$$

We see that events E_1 and E_2 do have two elements in common, hence they are not mutually exclusive events.

THEOREM 13.2. The probability of E_1 or E_2, where events E_1 and E_2 are not mutually exclusive is: $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

Now in this example

$$P(E_1 \cap E_2) = \frac{\text{No. of outcomes favorable to both } E_1 \text{ and } E_2}{\text{Total no. of possible outcomes}}$$

$$P(E_1 \cap E_2) = \frac{2}{36},$$

also

$$P(E_1) = \frac{\text{No. of outcomes in which the sum is greater than 9}}{\text{Total no. of possible outcomes}}$$

$$P(E_1) = \frac{6}{36},$$

and

$$P(E_2) = \frac{\text{No. of outcomes whose sum is odd, greater than 8}}{\text{Total no. of possible outcomes}}$$

$$= \frac{6}{36}.$$

Hence

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} \\ &= \frac{10}{36}. \end{aligned}$$

Exercises for Lesson 13.

1. Copy the following table, and then use the information in the table set up in this section to complete it:

Sum of 2 dice:	2	3	4	5	6	7	8	9	10	11	12
Number of ways:	1	___	___	___	___	6	___	___	___	2	___
Probability:	$\frac{1}{36}$	___	___	___	___	$\frac{1}{6}$	___	___	___	$\frac{1}{18}$	___

2. Add a fourth line to the table, showing the probabilities as percents. Each percent should be rounded off to the nearest whole number.

Percent probability: 3 ___ ___ ___ ___ 17 ___ ___ ___ 6 ___ .

3. In the dice game being played by the men in the alley, if a 4, 5, 6, 8, 9, or 10 is rolled, the person who rolls the dice does not win or lose immediately.

(a) How many ways are there altogether of getting any of these numbers?

(b) What is the probability that the person who rolls will neither win nor lose on the first throw?

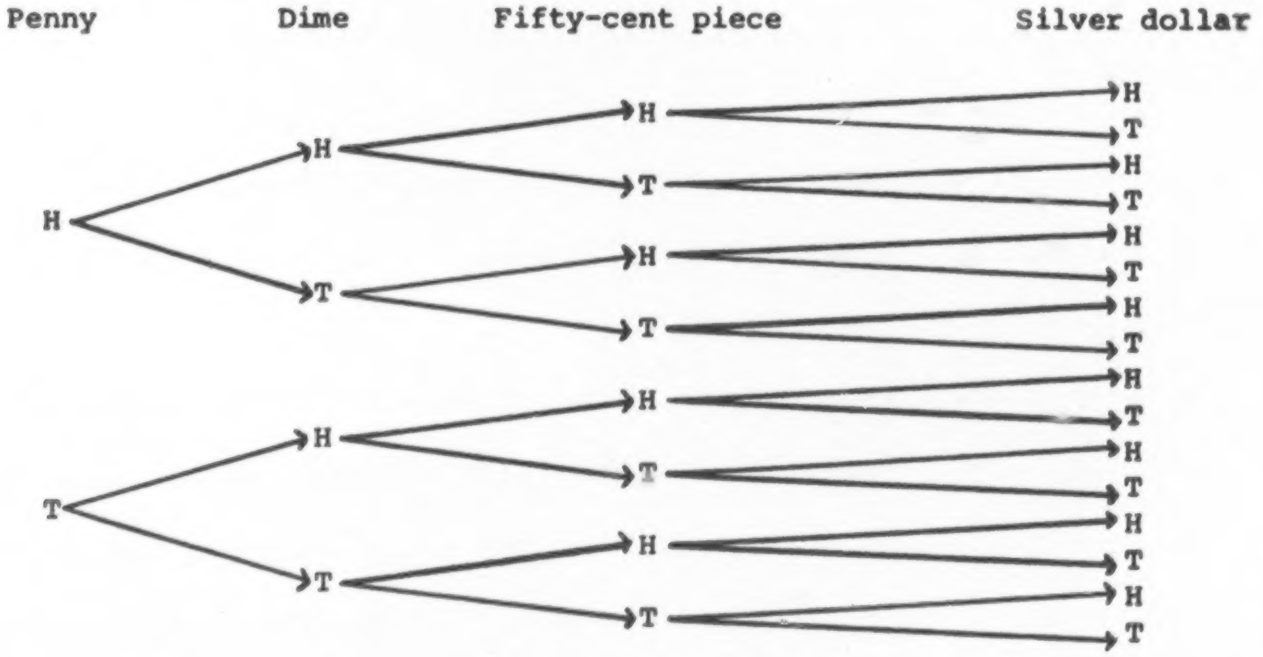
4. If the person rolling gets a 4, 5, 6, 8, 9, or 10, he must roll the same number again before rolling a 7 in order to win.

(a) Which of these numbers are least likely to be rolled?

(b) Which of these numbers are most likely to be rolled?

5. Suppose that a penny, a dime, a fifty cent piece, and a silver dollar are tossed at once.

(a) Using the Tree diagram which follows, list all possible outcomes:



(b) What is the probability of four heads?

(c) What is the probability of getting four tails?

(d) What is the probability of getting three tails and a head?

(e) What is the probability of getting two tails and two heads?

6. An opaque cookie jar contains ten cookies of which five are chocolate chip, three are ginger snaps, and two are vanilla wafers. All the cookies are identical in size, so each cookie is equally likely to be picked if you reach in the jar and take one cookie at random.

(a) What is the probability that you will pick a ginger snap cookie (which you don't like)?

(b) Assuming that you pick a ginger snap and return it to the jar (no one is watching), what is the probability you will get a ginger snap on the second draw?

(c) Suppose that you pick a ginger snap on the first draw and a vanilla wafer on the second draw. You return both cookies to the jar. What is the probability that you will pick a chocolate chip cookie on the third draw (which is really what you want)?

7. There are six snakes in a basket, 2 coral snakes and 4 garter snakes. Each snake is wrapped in a handkerchief. As you reach into the basket and unwrap a handkerchief, what is the probability that you will be bitten on the hand by a poisonous snake (assume all the snakes are equally likely to bite you).

8. Over a three year span an instructor has taught 800 students. During this period the grade "A" was given to 50 students and the grade "B" was given to 300 students. Suppose that you are given the complete roll of all students taught by this instructor. Suppose you randomly select a name from this roll. What is the probability that the student selected will have received an "A" or "B"?

9. Three job openings occur, and the employer decides to fill the openings by selecting at random from among four equally qualified applicants, W, X, Y, Z. What is the probability that both X and Y will be chosen or W and Z will be chosen?

Answers for Lesson 13.

1. and 2.

Sum of dice:	2	3	4	5	6	7	8	9	10	11	12
Number of ways:	1	2	3	4	5	6	5	4	3	2	1
Probability:	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
Percent Probability:	3	6	8	11	14	17	14	11	8	6	3

3. (a) 24 (b) $\frac{2}{3}$

4. (a) 4 and 10 (b) 6 and 8

5. (a) HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,
THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT

(b) $\frac{1}{16}$ (c) $\frac{1}{16}$ (d) $\frac{1}{4}$ (e) $\frac{3}{8}$

6. (a) $\frac{3}{10}$ (b) $\frac{3}{10}$ (c) $\frac{1}{2}$

7. $\frac{1}{3}$ 8. $\frac{7}{16}$ 9. 1

LESSON 14

Probability

A deck of playing cards contains 52 cards divided into four suits of thirteen cards each: clubs, hearts, spades and diamonds. The cards within each suit are: ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3, and 2.

Suppose the deck is carefully shuffled and you are asked to randomly draw one card from the deck. What is the probability that the card that you draw will be a spade? There are 13 ways of drawing a spade out of 52 ways a card could be drawn,

$$P(\text{SPADE}) = \frac{13}{52} = \frac{1}{4} ,$$

or you have one chance in four of drawing a spade. If the card which was drawn is put back into the deck and the deck is reshuffled, your chances of again drawing a spade are the same as before: 1 chance in 4. Since the card that was drawn first was put back into the deck before the second card was drawn, it has no influence on what happens in the second draw. Such events which have no influence on each other are said to be INDEPENDENT.

Now, let's change the problem slightly. Suppose the deck is carefully shuffled, you draw a card, note what it is, then return it to the deck. The deck is reshuffled and you draw once again. What are your chances of drawing two spades? These two events are independent and for such events the probability of event E_1 and E_2 occurring, $P(E_1 \cap E_2)$, is the product of the probability of E_1, $P(E_1)$, and the probability of E_2, $P(E_2)$.

THEOREM 14.1: If E_1 and E_2 are independent events then

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2).$$

For the problem of drawing two spades:

$$\begin{aligned} P(\text{getting a spade on first draw}) &= P(E_1) \\ &= \frac{13}{52} = \frac{1}{4}, \\ P(\text{getting a spade on second draw}) &= P(E_2) \\ &= \frac{13}{52} = \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} P(\text{spade on first draw and spade on second draw}) &= P(E_1 \cap E_2) \\ &= P(E_1) \cdot P(E_2) \\ &= \frac{1}{4} \cdot \frac{1}{4} \\ &= \frac{1}{16}. \end{aligned}$$

You have one chance in 16 of drawing two spades.

Now suppose that two cards are drawn from the deck in succession, the second without replacing the first. What is the probability that they will both be spades? It is apparent now that your first draw will affect what happens on the second draw. Events which are affected or influenced by other events are said to be DEPENDENT.

The probability of drawing two spades in succession without replacing the first card can be solved in the following way. The probability of drawing the first spade is $\frac{13}{52}$. Once the first card has been drawn, the probability that the second card will be a spade is not the same. Suppose that the first card is a spade, then there are 12 spades remaining in a deck of 51 cards. Thus the probability of drawing a second spade is $\frac{12}{51}$. And the probability of successively drawing two spades without replacement can be solved using the multiplication principle

$$\frac{13}{52} \cdot \frac{12}{51} = \frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17},$$

one chance in seventeen.

This method of solution can be extended to successive draws without replacement involving more than two cards. For example, what is the probability of drawing four aces?

$$\begin{aligned} P(\text{drawing 4 aces in succession}) &= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} \\ &= \frac{1}{13} \cdot \frac{1}{17} \cdot \frac{1}{25} \cdot \frac{1}{49} \\ &= \frac{1}{270{,}725}. \end{aligned}$$

Now let's consider a related problem. Suppose that you are going to randomly draw a card from a shuffled deck, what are the chances that you will not draw a spade? In order to see how to effectively handle this problem let's consider COMPLEMENTARY PROBABILITIES. Such probabilities are based on the idea that an event must either happen or not happen. You recall that the probability of an event which is certain to occur is one It is apparent then that an event will happen, E, or will not happen, E'. So

$$\begin{aligned} P(\text{happens or doesn't happen}) &= P(E \cup E') \\ &= P(E) + P(E') \\ &= 1. \end{aligned}$$

The probabilities that an event will happen and that it will not happen are called COMPLEMENTARY. Since the sum of two complementary probabilitie is always one, you can find either one by subtracting the other from one.

Previously we saw that the probability of the event E_1 "drawing a spade" was $\frac{13}{52}$, so the probability of the event E', "not drawing a spade" could be computed:

$$P(\text{not drawing a spade}) = 1 - P(\text{drawing a spade})$$

$$P(E') = 1 - P(E)$$

$$P(E') = 1 - \frac{13}{52}$$

$$P(E') = \frac{39}{52}\ .$$

A very interesting problem in probability, which rather betrays the intuition, can be solved using complementary probabilities. In a room containing fifty people (whom we will assume are not celebrating a common birthday!) what is the probability that at least two of them will have birthdays on the same day? Ignoring leap year, there are 365 days in a year, so it would appear quite remote that two such people would indeed be present.

So much for what "seems" to be true, let's analyze the problem. In order that we might more easily see how to construct our probability model, let's consider a room with fewer people, say two. Now we repeat the question. What is the probability that in a group of two people both will have birthdays on the same day? Let E be the event "2 birthdays on the same day". The first person considered can have his birthday on any day. The probability of the event E', "2 do not have the same birthday" is $\frac{364}{365}$. That is, once the first person's birthday is known, the second person has 364 chances out of 365 of not having that birthday.

Then

$$P(\text{2 have same birthday}) = 1 - P(\text{2 do not share birthday})$$

$$P(E) = 1 - P(E')$$

$$= 1 - \frac{364}{365}$$

$$= \frac{1}{365},$$

which is fairly small.

But let's expand our group to three people. What is the probability that out of three people, two will have the same birthday? The probability that the third person's birthday is not the same as either of the other two is $\frac{363}{365}$. Again, the idea is that once person one and two's birthday is known there are 363 possible days out of 365 for a birthday which will not coincide with the first two. We can now apply the multiplication principle used previously to obtain the probability that both the second and third person's birthday (which differ from each other) will be different from the first.

$$P(\text{2 share birthday}) = 1 - P(\text{2 do not share birthday})$$

where

$$P(\text{2 do not share birthday})$$

$$= P(\text{2 does not share with 1 } \underline{\text{and}} \text{ 3 does not share with 1 \& 2})$$

$$= P(\text{2 does not share with 1}) \cdot P(\text{3 does not share with 1 \& 2})$$

$$= \frac{364}{365} \cdot \frac{363}{365}.$$

So

$$P(\text{2 share birthday}) = 1 - P(\text{2 do not share birthday})$$

$$= 1 - \frac{364}{365} \cdot \frac{363}{365}$$

$$= 1 - 0.992$$

$$= 0.008.$$

Continuing in this way, the probability that two people in a group of 50 will share the same birthday is

P(2 share birthday) = 1 - P(2 do not share birthday).

P(2 do not share birthday) = P(2 does not share with 1 and 3 does not share with 1 & 2 and 4 does not share with 1 & 2 & 3 and · · · 50 does not share with 1 & 2 & 3 · · · & 49)

= P(2 does not share with 1) · P(3 does not share with 1 & 2) · · · P(50 does not share with 1 & 2 & 3 · · · & 49)

$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \quad \cdots \quad \cdot \frac{316}{365} .$$

So

$$\text{P(2 share birthday)} = 1 - \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \quad \cdots \quad \cdot \frac{316}{365}$$

$$\doteq 1 - 0.0296$$

$$= 0.9704.$$

There is a 97% chance that out of a random group of 50 people, two will have the same birthday.

Exercises for Lesson 14.

1. Determine which of the following pairs of events are independent.

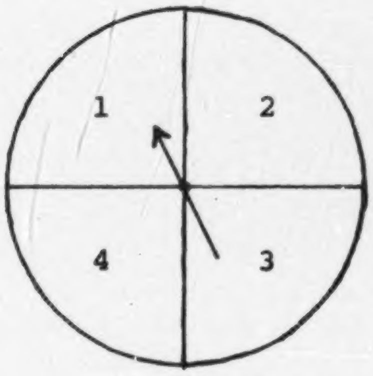
(a) A 1 on the first roll of a die, and a 3 on the second.

(b) 5 on the toss of a die, heads on the toss of a nickel.

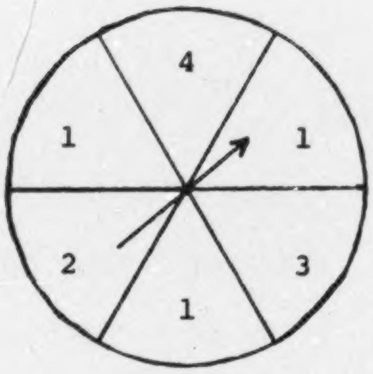
(c) From a cookie jar containing 3 ginger snaps, 2 chocolate cookies, and 5 lemon snaps, first drawing a ginger snap, then drawing a lemon snap on the second draw.

2. What is the probability of drawing two queens from a deck of cards if one card is drawn then replaced before the second is drawn?

3. Given the following wheels:



Wheel A



Wheel B

(a) What is the probability that both will stop on the 1?

(b) What is the probability that both will stop on the 2?

(c) What is the probability that the pointer on A will stop on 4 and the pointer on B will stop on 3?

(d) What is the probability that the pointer on A will stop on 1 and the pointer on B will stop on 4?

4. On a basketball team, Jim has a free throw shooting average of 0.75 and Tom has a free throw shooting average of 0.90. Both players have a chance to shoot free throws.

(a) What is the probability that Jim and Tom will both hit their shots?

(b) What is the probability that Jim or Tom or both will hit their shots?

5. Suppose there are 3 marbles in a box: one red, one white, one black.

(a) Suppose a marble is drawn, replaced, and a second marble drawn. What is the probability of a red on the first draw and white on the second?

(b) Now suppose a marble is drawn, then a second marble is drawn without replacing the first marble. What is the probability of red on the first draw and black on the second?

6. Suppose a pair of dice are thrown twice, what is the probability that

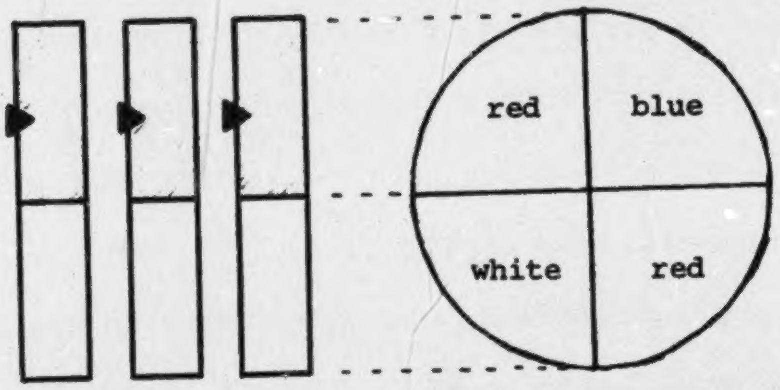
you will get:

(a) a pair of 6's each time?

(b) a sum of 7 each time?

(c) a sum of 7 the first time and a sum of 11 the second time?

7. Suppose we have a machine that has three disks mounted side by side:



and suppose the disks are divided into four equal parts and each part is painted according to the color scheme indicated. Suppose the disks are spun independently of each other and gradually come to a stop with some color under the indicators. Find the probability of getting the colors red, white, and blue, in that order.

8. Find the probability of getting all red.

9. Find the probability of getting all white.

10. Suppose that you make successive draws from a deck of 52 cards without replacement, what is the probability of drawing 2 queens?

11. What is the probability of drawing the jack of diamonds, followed by the queen of spades?

12. In terms of colors the deck is made up of two colors. The hearts and diamonds are red and the spades and clubs are black. What is the probability of drawing 3 black cards?

13. All kings, queens, and jacks are called face cards. What is the probability of drawing 4 face cards without replacement?

14. A flush is a poker hand of 5 cards all of the same suit. When being dealt a flush, it doesn't matter which card you get first. So the probability of getting the first card is $\frac{52}{52}$. After the first card has been dealt, there are only 12 cards left in the deck that are of the same suit. What is the probability that if five cards were dealt out, the hand would be a flush?

15. A "Yarborough" is a hand of 13 cards that contains no aces, kings, queens, or jacks. Find the probability that a "Yarborough" will be dealt. (Hint: Since 16 cards can't appear in the hand, the first card dealt can be any one of 36 cards.)

16. Which is more probable, a flush in poker or a "Yarborough" in bridge?

17. Using the method given in this lesson, compute the probability that 2 people in a group of 5 have the same birthday.

18. As the number of people in a group increases what happens to the probability that two people in the group will share the same birthday?

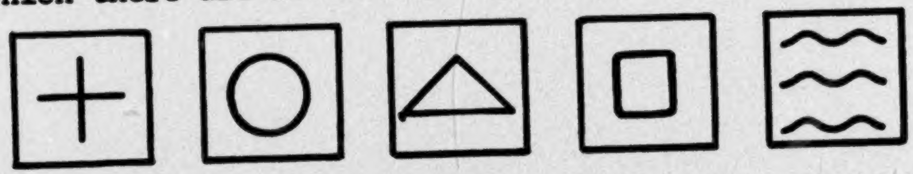
19. If a pair of dice are rolled,

    (a) What is the probability that they will come up 11?

    (b) What is the probability that they will not come up 11?

20. If a single die is rolled once, what is the probability that it will not come up 6?

21. To test for extrasensory perception (ESP) a special deck of 25 cards is used in which there are five cards each containing the symbols:



If the deck is shuffled and a card drawn, what is the probability that someone could guess the symbol by pure luck?

22. What is the probability that someone could guess 4 cards in succession, if each card is returned to the deck and the deck is reshuffled before each new card is drawn?

23. Suppose a card containing each of the symbols is taken from the deck and the five cards arranged face down side by side in random order, what is the probability of guessing the order by luck? (Hint: the events of guessing the cards are dependent. The probabilities for the first 2 cards are $\frac{1}{5}$ and $\frac{1}{4}$.)

Answers for Lesson 14.

1. (a), (b) 2. $\frac{1}{169}$

3. (a) $\frac{1}{8}$ (b) $\frac{1}{24}$ (c) $\frac{1}{24}$ (d) $\frac{1}{24}$

4. (a) $\frac{27}{40}$ (b) $\frac{39}{40}$ 5. (a) $\frac{1}{9}$ (b) $\frac{1}{6}$

6. (a) $\frac{1}{1296}$ (b) $\frac{1}{36}$ (c) $\frac{1}{108}$

7. $\frac{1}{32}$ 8. $\frac{1}{8}$ 9. $\frac{1}{64}$ 10. $\frac{1}{221}$ 11. $\frac{1}{2652}$

12. $\frac{6}{51}$ 13. $\frac{297}{162435} = \frac{99}{54145}$

14. $\frac{33}{16660} \approx .2\%$ 15. $\frac{631098}{173428073} \approx .36\%$

16. Yarborough 17. .0271 18. Probability increases

19. (a) $\frac{1}{18}$ (b) $\frac{17}{18}$

20. $\frac{5}{6}$ 21. $\frac{1}{5}$ 22. $\frac{1}{625}$ 23. $\frac{1}{120}$